BORSUK-ULAM TYPE THEOREMS FOR PROPER Z_p-ACTIONS ON (MOD p HOMOLOGY) n-SPHERES

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0. Introduction.

In 1933 Borsuk [1] proved the following

**BORSUK-ULAM theorem.** For any map \( f: S^n \to R^n \) there is an \( x \in S^n \) such that \( f(x) = f(-x) \).

In 1955 Bourgin and (independently) Yang published proofs of the following generalization (see [2], [3], and [16]).

**BOURGIN-YANG theorem.** For any map \( f: S^n \to R^k \) the covering dimension of

\[
A(f) = \{ x \in S^n \mid f(x) = f(-x) \}
\]

is at least \( n - k \).

In 1960 Conner and Floyd generalized the result further (see [7] and [8]). They proved what here will be called

**CONNER-FLOYD theorem.** Let \( T \) be a fixed point free, differentiable involution on the n-sphere \( S^n \), and let \( f: S^n \to M^k \) be a continuous map into a differentiable k-manifold \( M^k \). Suppose that

\[
(0.1) \quad f_\ast = 0 : H_n(S^n; Z_2) \to H_n(M^k; Z_2).
\]

Then the covering dimension of

\[
A(T,f) = \{ x \in S^n \mid f(x) = f(xT) \}
\]

is at least \( n - k \).

In [8] Conner and Floyd ask the following questions:

1. Can all differentiability hypotheses be eliminated?
2. Can \( S^n \) be replaced by a closed n-manifold which is a mod 2 homology n-sphere?
3. Can \( M^k \) be replaced by non-manifolds?

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It also seems natural to ask the following

4. **Question.** Let $G$ be a finite group of order $|G| > 2$ acting properly on the $n$-sphere $S^n$ via the map $\mu : S^n \times G \to S^n$. For any map $f : S^n \to M^m$ into a compact, topological $m$-manifold $M^m$ let

$$A(\mu ; f) = \{ x \in S^n \mid f(x) = f(xg) \text{ for all } g \in G \}.$$ 

Is the covering dimension of $A(\mu ; f)$ necessarily \( \geq n - (|G| - 1)m \)?

**Remarks.** A condition analogous to (0.1) does not appear because for $n \geq (|G| - 1)m$ and $|G| \geq 3$ it would automatically be true (and for $n < (|G| - 1)m$ there is no question).

Covering dimension (abbreviated cov. dim.) is taken in the sense of [14], say; it does not really matter in which sense it is taken since cohomological dimension (see [6]) rather than covdim will be used.

In this paper, questions 1, 2, and 4 are treated. The (partial) answers obtained are:

**Question 1:** Yes.

**Question 2:** For the Bourgin-Yang theorem (and certain other cases): Yes.

**Question 4:** For $G = Z_p$, $p$ prime, and with $Z_p$-orientability of $M^m$: Yes (see also the Note at the end of this section).

**Question 1** was already considered in [12].

More precisely, a theorem (see section 4) which has the following two corollaries will be proved.

**Mod $p$ Conner-Floyd Theorem.** Let $\mu : S^n \times Z_p \to S^n$ be a proper action of the cyclic group of prime order $p$ on the $n$-sphere. Consider a map $f : S^n \to M^m$ into a compact, topological $m$-manifold $M^m$. If $p = 2$, assume that

$$f_* = 0 : H_n(S^n ; Z_2) \to H_n(M^m ; Z_2),$$

and if $p$ is odd, assume that $M^m$ is $Z_p$-orientable. Then the cohomological dimension (with coefficients $Z_p$) of

$$A(\mu ; f) = \{ x \in S^n \mid f(x) = f(xg) \text{ for all } g \in Z_p \}$$

is at least $n - (p - 1)m$.

**Mod $p$ Bourgin-Yang Theorem.** Let $S^n$ be a closed $n$-manifold which is a mod $p$ homology $n$-sphere. Let $\mu : S^n \times Z_p \to S^n$ be a proper action.
of $Z_p$. Then for any map $f : S^n \to R^n$ the cohomological dimension (coefficients $Z_p$) of

$$A(\mu ; f) = \{ x \in S^n \mid f(x) = f(xg) \text{ for all } g \in Z_p \}$$

is at least $n - (p - 1)m$.

The proof will be based upon ideas dating back to Yang [16]; they were also used by Conner and Floyd in [8].

**Note** (added just before printing). Using obstruction theory it is easy to prove the following:

Let $k$ be an odd, non-prime number $\neq 9$ and let $\mu : S^k \times Z_k \to S^k$ be the standard action. Then there exists a map $f : S^k \to R$ such that $A(\mu ; f) = \emptyset$.

It seems possible to obtain some positive results (for maps $S^n \to R$, $n$ large, and $Z_k$-action on $S^n$) by using $K$-theory characteristic classes. I hope to return to this in a future publication.

1. **Notation.**

Let $G$ be a finite group of order $|G| = k + 1$. If $|G|$ is even, let $q = 2$, and if $|G|$ is odd, let $q$ be an arbitrary prime. By $H_\ast, H^\ast, \overline{H}^\ast$ we denote singular homology, singular cohomology and Alexander–Spanier cohomology, respectively; if no coefficients are mentioned, $Z_q$ is understood. For Alexander–Spanier cohomology we shall freely change between the two definitions given by Spanier (p. 289 and p. 308 of [15]). By $\iota : \overline{H}^\ast \to H^\ast$ we denote the natural transformation given on p. 289 of [15].

The word *manifold* will be taken to mean a $Z_q$-orientable topological manifold; *cc-manifold* will mean a closed (that is, compact and without boundary) and connected manifold. For any compact pair $(A, B)$ in a $Z_q$-orientable $n$-manifold $M$, there is the Alexander–Spanier duality isomorphism

$$\overline{\gamma} : H_\ast(M - B, M - A) \to H^{n - \ast}(A, B),$$

defined via the slant product as in [15].

A *G-space* $X$ will mean a space $X$ together with a map $\mu : X \times G \to X$ (written $\mu(x, g) = xg$) such that $x(gh) = (xg)h$ and $x1 = x$. As usual $\mu$ (or $X$) is called *proper* if

$$(\exists x : xg = x) \Rightarrow g = 1.$$ 

For G-spaces $X_1$ and $X_2$ let $X_1 \times_G X_2$ be the quotient space $(X_1 \times X_2)/G$ where $G$ acts diagonally on $X_1 \times X_2$. If $Y$ is any space, let $YG$ be the
product of $|G|$ copies of $Y$; writing its elements as $\sum_g y(g)g$ it is a $G$-space under the action

$$(\sum y(g)g)h = \sum y(g)gh = \sum y(gh^{-1})g.$$  

In $YG$ there is the $G$-invariant (and $G$-trivial) subspace

$$\Delta Y = \{\sum y(g)g \mid y(g) = y(1) \text{ for all } g \in G\}$$

(the "diagonal"). If $y_0$ is a base point of $Y$, then $\Delta Y$ and $YG$ receive $\ast = \sum y_0g$ as a base point.

Fix a ce-manifold $\mathcal{S}^n$ which is a mod$q$ homology $n$-sphere, and let $\mu: \mathcal{S}^n \times G \to \mathcal{S}^n$ be a proper $\mathbb{Z}_q$-orientation-preserving action of $G$ upon $\mathcal{S}^n$. For any map $f: \mathcal{S}^n \to Y$ put

$$A(\mu; f) = \{x \in \mathcal{S}^n \mid f(x) = f(xg) \text{ for all } g \in G\}.$$  

Associated with $\mu$ there is a vector bundle $\xi_\mu$ described as follows. Take $IG$ to be the augmentation ideal of the group algebra $RG$, that is, $IG$ consists of those $\sum r(g)g \in RG$ for which $\sum r(g) = 0$. Now $\xi_\mu$ is obtained by screwing in $IG$ as fibre in the principal $G$-bundle $\mathcal{S}^n \to \mathcal{S}^n/G$, that is,

$$\xi_\mu = (\mathcal{S}^n \times_G IG \to \mathcal{S}^n/G).$$

It is easily seen that $\xi_\mu$ is $\mathbb{Z}_q$-orientable. Hence $\xi_\mu$ has a (mod$q$) Euler class

$$e_q(\xi_\mu) \in H^k(\mathcal{S}^n/G).$$

As noticed in the introduction, cohomological dimension with coefficients $A$ ($A$ being some abelian group) is taken in the sense of [6]; it will be abbreviated $\text{cd}(\cdot; A)$. If $X$ is a compact, proper $G$-space, then

$$\text{cd}(X; A) = \text{cd}(X/G; A).$$

To see this, cover $X$ by closed subsets $X_\varepsilon$ chosen so small that the projection $X \to X/G$ gives a homeomorphism when restricted to $X_\varepsilon$. The above equality then follows immediately from the sum-theorem and the monotonicity property of $\text{cd}(\cdot; A)$ (see [6], theorem 4.1 and lemma 2.2 together with remark 2.11). The inequality

$$\text{cd}(X; A) \leq \text{covdim}(X)$$

(see [6]) shows that it is actually better to work with $\text{cd}$ than with $\text{covdim}$.

Finally a map $f: \mathcal{S}^n \to Y$ is called nice provided the following holds: There is a map $f_0: \mathcal{S}^n \to Y$ and a point $y_0 \in Y$ such that
(1.1) \[ f_0 \simeq f, \]
where \( \simeq \) means "homotopic to", and

(1.2) \[ \forall x \in S^n : f_0(xy) \neq y_0 \text{ for at most one } g \in G. \]

Notice that any map \( f : S^n \to Y \) is nice under either of the following conditions:

(1.3) \[ S^n = S^n, \]

(1.4) \[ Y \text{ is contractible (especially for } Y = R^m). \]

2. The main proposition.

2.1. Proposition. Let \( f : S^n \to M^m \) be a map into a compact \( m \)-manifold \( M^m \). Suppose that

(2.1) \[ \text{cd}(A(\mu;f)/G;Z_q) < n - km, \]
(2.2) \[ f \text{ is nice}, \]
(2.3) \[ f_* = 0 : H_n(S^n) \to H_n(M^m). \]

Then

(2.4) \[ e_q(x\mu)^m = 0. \]

Remarks. 1. If \( n < km \), then \( e_q(x\mu)^m \in H^{mk}(S^n/G) = 0 \); therefore, assume from now on that \( n \geq km \).

2. With \( n \geq km \) condition (2.3) is trivially fulfilled unless \( G = Z_2 \) and \( m = n \) (of course I do not want to consider \( G = 1 \), by the way I also tacitly assume \( n > 0, m > 0 \)).

3. One may, and we do take \( M^m \) a cc-manifold. In fact, if \( M^m \) is not connected one just has to regard \( f \) as a map into the relevant component of \( M^m \). And if \( M^m \) has a non-empty boundary one considers \( f \) as a map into \( DM^m \). \( (DM^m = \text{"the double of } M^m \text{"}) \) consists of two copies of \( M^m \) identified along their boundaries. \( DM^m \) is a manifold because the boundary of \( M^m \) is collared in \( M^m \), see [4].

4. The assumption (2.1) implies that

(2.1') \[ H^{n-km}(A(\mu,f)/G) = 0. \]

This weaker assumption is the one that is actually used in the proof.

Digression 1. Before turning to the proof of the proposition, we consider the special case of maps \( f : S^n \to R^m \) (to get this as a special case notice that \( f(S^n) \) is contained in a sufficiently big closed disc in \( R^m \)). There is the \( |G|m \)-dimensional \( Z_q \)-orientable vector bundle
\( \eta = (\mathcal{S}^n \times_G R^m G \to \mathcal{S}^n/G) \)

containing the trivial \( |G| \)-dimensional vector bundle
\( \varepsilon = (\mathcal{S}^n \times_G \Delta R^m \to \mathcal{S}^n/G) \)
as a subbundle. The quotient \( \eta/\varepsilon \) is easily seen to be
\[ m\xi_{\mu} = \xi_{\mu} \oplus \ldots \oplus \xi_{\mu} \]
(the direct sum of \( m \) copies of \( \xi_{\mu} \)). Hence there is a short exact sequence of vector bundles

\[
\begin{array}{ccc}
\mathcal{S}^n \times_G \Delta R^m & \longrightarrow & \mathcal{S}^n \times_G R^m G \\
\varepsilon & \downarrow & \eta \\
\mathcal{S}^n/G & \leftarrow & m\xi_{\mu}
\end{array}
\]

The map \( f \) induces a cross-section \( s \) of \( \eta \), namely
\[ s(xG) = (x, \sum f(xy^{-1})g) G, \]
and
\[ s^{-1}(\mathcal{S}^n \times_G \Delta R^m) = A(\mu; f)/G. \]

This means that \( \pi s \) is a cross-section of \( m\xi_{\mu} \), which is zero only above \( A(\mu; f)/G \). Hence the \((\text{mod} q)\) Euler class of \( m\xi_{\mu} \) satisfies
\[ (2.5) \quad e_q(\xi_{\mu})^m \in \text{Im} \left[ H^{kn}(\mathcal{S}^n/G, \mathcal{S}^n/G - A(\mu, f)/G) \to H^{km}(\mathcal{S}^n/G) \right]. \]

By definition of \( H, (2.1') \) means that
\[ \lim_{U} H^{n-km}(U) = 0 \]
as \( U \) ranges over all open neighbourhoods of \( A(\mu; f)/G \) in \( \mathcal{S}^n/G \). For any \( x \in H^{n-km}(\mathcal{S}^n/G) \) one can then find an open \( U_x \supseteq A(\mu; f)/G \) such that
\[ (2.6) \quad x \in \text{Im} [H^{n-km}(\mathcal{S}^n/G, U_x) \to H^{n-km}(\mathcal{S}^n/G)]. \]

From (2.5) and (2.6) follows that the cup-product-map
\[ (2.7) \quad e_q(\xi_{\mu})^m \cup - : H^{n-km}(\mathcal{S}^n/G) \to H^n(\mathcal{S}^n/G) \]
vanishes. In a \((\mathbb{Z}_q\text{-orientable})\) manifold, however, the cup-product-pairing to the top-dimension is non-singular (using field coefficients). Hence (2.7) implies that \( e_q(\xi_{\mu})^m = 0 \).

The proof of the proposition is divided into 3 lemmas. The first one of these gives a condition for the vanishing of \( e_q(\xi_{\mu})^m \) in terms of \( \mathcal{S}^n \) and \( M^m \). It is motivated by the following
DIGRESSION 2. Suppose for a moment that \( S^n, \mu, \) and \( M^m \) are differentiable. There are the imbeddings
\[
S^n \times_G \ast \subseteq S^n \times_G \Delta M \subseteq S^n \times_G MG,
\]
where \( \ast \) is the basepoint of \( MG \) corresponding to some basepoint of \( M \). Also \( S^n \times_G \ast \) is identified with \( S^n/G \) in a canonical way, and it is not hard to see that under this identification
\[
i^*v = m\xi_\mu,
\]
where \( v \) is the normal bundle of the imbedding \( j \).

Consider then the commutative diagram (for any \( G \)-space \( X \) let \( \overline{X} = S^n \times_G X \))
\[
\begin{array}{ccc}
H_{n+m}(MG) & \stackrel{j^*}{\rightarrow} & H_{n+m}(\overline{\Delta M}) \stackrel{j^*}{\rightarrow} H_{n+m}(MG, \overline{MG-\ast}) \\
\cong \downarrow \varphi & & \cong \downarrow \varphi \\
H^{km}(MG) & \downarrow j^* & H^{km}(\overline{\ast}) \\
& \downarrow i^* & \\
H^{km}(\overline{\Delta M}) & \rightarrow & H^{km}(\overline{\ast})
\end{array}
\]
It is well known (see, for example, [11]) that the (mod \( q \)) Euler class of \( v \) is
\[
e_q(v) = j^*(\varphi j)_* \sigma,
\]
where \( \sigma \) is the orientation class of \( \overline{\Delta M} \). Hence \( e_q(\xi_\mu)^m = i^* e_q(v) \) vanishes if and only if the inclusion
\[
j' : \Delta M \rightarrow (MG, MG-\ast)
\]
has
(2.8) \[H_{n+m}(j') = 0.\]

The first lemma states that this is still true if all differentiability hypotheses are dropped.

2.2. LEMMA. (2.4) and (2.8) are equivalent.

PROOF. Let \( D_1 \) be an open disc around \( m_0 \) in \( M \), and let \( j'' \) be the inclusion
\[
j'' : \overline{\Delta M} \rightarrow (MG, MG-D_1G).
\]
It is an easy consequence of “compact support” (see lemma 12, p. 204 of [15]) that (with \( D_1 \) sufficiently small) (2.8) is equivalent to
(2.9) \[H_{n+m}(j'') = 0.\]
Then look at the commutative diagram below. Here $D_2$ is a closed disc around $m_0$ in $M$ with $D_1$ contained in the interior of $D_2$, $D$ is the closure of $D_1$, $\partial D$ the boundary of $D$ and $(DG)^\dagger$ the boundary of $DG$. The map $j''''$ is the inclusion of $\overline{\Delta(\partial D)}$ in $(DG)^\dagger$, and $a, b, c, d$ denote maps induced by inclusion.

The map $a$ is monic, because $\overline{\Delta(M - D_1)}$ is an $(n+m)$-manifold with boundary (so that $H_{n+m}(\Delta(M - D_1)) = 0$). The isomorphisms labelled $\text{exc}$ are given by excising $\overline{\Delta M - \Delta D_2}$ and $\overline{MG - \Delta G}$, respectively. By $\partial$ we denote boundaries; they are monic because

$$H_{n+m}(\Delta D_2) = H_{n+m}(D_2 G) = 0$$

$(\Delta D_2$ and $D_2 G$ are $G$-equivariantly contractible so that $\overline{\Delta D_2}$ and $\overline{D_2 G}$ are homotopy-equivalent to $\ast = \mathcal{P}n|G)$. The deformation retractions (defretr) referred to are obtained as follows. Take $D_i$ to be a disc in $R^m$ of radius $i$ ($i = 1, 2$); the formula

$$\tilde{g}(x, \sum d(g)g, t) = (x, \sum \min \{1, (||d(g)||^{-1} - 1)t + 1\} d(g)g)$$

defines a map

$$\tilde{g} : \mathcal{P}n \times (D_2 G - D_1 G) \times I \to \mathcal{P}n \times (D_2 G - D_1 G).$$

Since, for each $t$, $\tilde{g}(\cdot, \cdot, t)$ is $G$-equivariant, there is an induced map

$$\varrho : (D_2 G - D_1 G) \times I \to D_2 G - D_1 G;$$

it is easily seen that $\varrho$ gives deformation retractions from $D_2 G - D_1 G$ onto $(DG)^\dagger$ and from $\overline{\Delta(D_2 - D_1)}$ onto $\overline{\Delta(D)}$.

If (in the diagram) $j_*'''' = 0$, then $\text{ker} c \neq 0$; it follows that $\text{ker} j_*'''\neq 0$. 
Since $H_{n+m-1}(\overline{\Delta(D)}) = \mathbb{Z}_q$, this implies that $j^{*'''} = 0$. Conversely, it is obvious that $j^{*'''} = 0$ implies $j^{*''} = 0$. This means that (2.9) is equivalent to

\begin{equation}
H_{n+m-1}(j^{*''}) = 0.
\end{equation}

Of course (2.10) is equivalent to

\begin{equation}
H^{n+m-1}(j^{*''}) = 0.
\end{equation}

Now recall the vector bundles $\varepsilon$, $\eta$, and $\xi_\mu$. Clearly the (total spaces of the) sphere bundles associated with $\varepsilon$ and $\eta$ may be taken as

\begin{align*}
S(\varepsilon) &= \overline{\Delta(D)} = \mathcal{S}^n \times_G \overline{\Delta(D)}, \\
S(\eta) &= \overline{DG} = \mathcal{S}^n \times_G (DG'),
\end{align*}

and with these identifications $j^{*'''}$ becomes the inclusion $S(\varepsilon) \subseteq S(\eta)$. There is then the commutative diagram

\begin{equation*}
\begin{array}{ccc}
H^{n+m-1}(S(\varepsilon)) & \xrightarrow{\delta} & H^{n+m-1}(B(\varepsilon), S(\varepsilon)) \\
\downarrow (j^{*'''}*) & & \downarrow \varphi \\
H^{n+m-1}(S(\eta)) & \xrightarrow{\delta'} & H^{n+m-1}(B(\eta), S(\eta))
\end{array}
\end{equation*}

Here $B(\varepsilon)$ and $B(\eta)$ are the ball bundles. The coboundaries $\delta$ and $\delta'$ are iso and epic, respectively (use the long exact sequences). By $\phi$ and $\phi'$ we denote Thom isomorphisms. Commutativity (at any rate up to sign) of the right hand square follows from a direct computation using three facts, namely:

1) $\varepsilon \otimes m\xi_\mu = \eta,$

2) Thom classes are multiplicative,

3) $e_q(\xi_\mu)^m$ may be described as the image of the Thom class of $m\xi_\mu$ under the composition

\begin{equation*}
H^{km}(B(m\xi_\mu), S(m\xi_\mu)) \xrightarrow{(\text{proj}^*)^{-1}} H^{km}(\mathcal{S}^n/G) \xrightarrow{\phi} H^{km}(\mathcal{S}^n/G).
\end{equation*}

Recalling that the cup-product pairing to the top dimension in $H^*(\mathcal{S}^n/G)$ is non-singular one reads off from the diagram that

\begin{equation}
e_q(\xi_\mu)^m = 0
\end{equation}

is equivalent to

\begin{equation}
H^{n+m-1}(j^{*''}) = 0
\end{equation}

This finishes the proof of lemma 2.2.
This lemma ties \( e_q(\xi_\mu)^m \) up with the manifolds (\( S^n \) and \( M^m \)) in question. The next thing to do is to bring the given map \( f: S^n \to M^m \) into the picture. This is done by introducing the map \( s: S^n/G \to \overline{MG} \), defined by the formula
\[
s(xG) = (x, \sum f(xg^{-1})g)G.\]
Observe that \( s \) is a cross-section in the fiber bundle
\[
(\overline{MG} = S^n \times_G MG \to S^n/G).\]
Also notice that \( s \) is a homeomorphism onto its image and that
\[
s^{-1}(\overline{AM}) = A(\mu; f)/G.\]

**Lemma 2.3.** Let \( f \) satisfy (2.2) and (2.3); then
\[
e_q(\xi_\mu)^m = 0
\]
if and only if the composition
\[
(2.12) \quad H_{n+m}(\overline{AM}) \overset{j^*}{\to} H_{n+m}(\overline{MG}) \overset{\cong}{\sim} H^{km}(\overline{MG}) \overset{s^*}{\to} H^{km}(S^n/G)
\]
vanishes.

**Proof.** In view of lemma 2.2 it suffices to prove that the diagram
\[
\begin{array}{cccccc}
H_{n+m}(\overline{AM}) & \overset{j^*}{\to} & H_{n+m}(\overline{MG}) & \overset{\cong}{\sim} & H^{km}(\overline{MG}) & \overset{s^*}{\to} H^{km}(S^n/G) \\
\downarrow j'^* & & & & \downarrow (ji)^* & \\
H_{n+m}(\overline{MG, MG - \star}) & \overset{\cong}{\sim} & H^{km}(\overline{\star}) & \overset{1^*}{\sim} \end{array}
\]
commutes (recall that \( \overline{\star} \) has been identified with \( S^n/G \)). If the dotted arrow is filled in by \( (ji)^* \), then the rectangle commutes, so it suffices to prove the triangle commutative, and that is where (2.2) and (2.3) come in. By (2.2) there is a point \( m_0 \in M \) and a map \( f_0: S^n \to M \) such that \( f \cong f_0 \) and, for each \( x \in S^n, f_0(xg) \neq m_0 \) for at most one \( g \in G \). It follows that \( s \) is homotopic to a cross section \( s_0: S^n/G \to \overline{MG} \) which factors through \( (\overline{MG})_1 \) where
\[
(\overline{MG})_1 = \{ \sum m(g)g \mid \text{at most one } m(g) \neq m_0 \} \subseteq MG.
\]
There results a commutative diagram
in which $i_1, i_2$ (and $ji$) are inclusions, $s_1$ is a factorization of $s_0$, and $p$ is induced by the projection $\mathcal{S}^n \times (MG)_1 \to \mathcal{S}^n$.

If $x \in H^{km}(MG)$, put $x' = p^*(ji)^*x - i_2^*x$; then

$$
(2.13) \quad i_1^*x' = 0 \quad \text{and} \quad s_1^*x' = (ji)^*x - s^*x.
$$

Hence commutativity of the triangle in question will follow from

$$
(2.14) \quad i_1^*x' = 0 \quad \text{implies} \quad s_1^*x' = 0.
$$

To prove (2.14) two cases are considered.

Case 1. $|G| > 2$. There is a relative homeomorphism

$$
\alpha : \mathcal{S}^n \times (M, m_0) \to (\overline{MG}_1, *)
$$

given by the formula

$$
\alpha(x, m) = (x, m \cdot 1 + \sum_{g+1} m_0 g) G.
$$

From this and the Künneth formula one gets

$$
H^{km}(\overline{MG}_1, *) = H^{km}(M, m_0) \oplus H^{km-n}(M, m_0) = 0.
$$

Therefore, the $i_1^*$ appearing in (2.14) is monic. But then (2.14) is trivially true.

Case 2. $G = \mathbb{Z}_2$. Here $k = 1$, so $H^{km}(M, m_0) \neq 0$, and the above argument does not work. Instead one looks at the following commutative diagram

$$
\begin{array}{c}
H^m((MZ_2)_1, *) \xrightarrow{j_1^*} H^m((MZ_2)_1) \xrightarrow{i_1^*} H^m(*) \\
\uparrow p_2^* \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow s_1^* \\
H^m((MZ_2)_1/Z_2, *) \xrightarrow{F^*} H^m(\mathcal{S}^n/Z_2)
\end{array}
$$

Here $p_2$ is induced by the projection $\mathcal{S}^n \times (MZ_2)_1 \to (MZ_2)_1$ and $F$ is induced by $\overline{F} : \mathcal{S}^n \to (MZ_2)_1$, where
Precisely as in [8, pp. 87–88] or in [12, proof of lemma 2.3] it is shown that $p_2^*$ is an isomorphism and that $F^* = 0$. But then $s_1^*j_1^* = 0$ and (2.14) follows by exactness of the row.

The final step in the proof of proposition 2.1 is

**Lemma 2.4.** If $f$ satisfies (2.1'), then (2.12) holds.

**Proof.** Let $\sigma$ be the orientation class of $\overline{\Delta M}$ and put

$$\varphi = s^*(i\gamma)_{j*} \sigma.$$  

By the so-often-referred-to non-singularity of the cup-product-pairing to $H^n(\mathcal{S}^n/G)$ it suffices to prove that

$$\varphi \cup - : H^{n-km}(\mathcal{S}^n/G) \to H^n(\mathcal{S}^n/G)$$

vanishes. Hence let $x \in H^{n-km}(\mathcal{S}^n/G)$. As in digression 1 there is an open $U_x \supseteq A(\mu; f)/G$ such that

(2.6) $$x \in \text{Im}[H^{n-km}(\mathcal{S}^n/G, U_x) \to H^{n-km}(\mathcal{S}^n/G)]$$

holds. Now choose a closed neighbourhood $V$ of $A(\mu; f)/G$ with $V \subseteq U_x$. There is then a neighbourhood $W$ of $\overline{\Delta M}$ (in $\overline{MG}$) with $s^{-1}(W) \subseteq V$. Thus $s$ gives a map

$$(\mathcal{S}^n/G, \mathcal{S}^n/G - V) \to (\overline{MG}, \overline{MG} - W),$$

and one gets a commutative diagram

$$
\begin{array}{ccc}
H_{n+m}(W) & \xrightarrow{i\gamma} & H^{km}(\overline{MG}, \overline{MG} - W) \\
\downarrow j^* & & \downarrow s^* \\
H_{n+m}(\overline{\Delta M}) & \xrightarrow{i\gamma} & H^{km}(\overline{MG}) \\
\end{array}
$$

from which it is seen that

(2.15) $$\varphi \in \text{Im}[H^{km}(\mathcal{S}^n/G, \mathcal{S}^n/G - V) \to H^{km}(\mathcal{S}^n/G)].$$

Since

$$(\mathcal{S}^n/G - V) \cup U_x = \mathcal{S}^n/G,$$

(2.6) and (2.15) imply that $\varphi \cup x = 0$ as desired.

Hereby proposition 2.1 is proved. Borsuk–Ulam-type theorems can be derived from it by computing $e_\mu(\xi_\mu)^m$. This is done in the next section.
3. Computation of $e_q(\xi_\mu)^m$.

In this section $e_q(\xi_\mu)^m$ is computed for all proper actions $\mu: S^n \times G \to S^n$ of some $G$ upon $S^n$ and for all values of $m$. For actions $\mu: \mathcal{F}^n \times G \to \mathcal{F}^n$ only partial results have been obtained.

**Proposition 3.1.** In the following cases, $e_q(\xi_\mu)^m \neq 0$:

\begin{align*}
(3.1) & \quad G = Z_q \quad \text{and} \quad n \geq (q-1)m, \\
(3.2) & \quad G = Z_4, \quad m = 1, \quad \text{and} \quad n \geq 3.
\end{align*}

If $G$ is periodic (hence especially if $\mathcal{F}^n = S^n$), then there are no other cases with $e_q(\xi_\mu)^m \neq 0$.

**Proof.** This is divided into small steps; it is always assumed that $n \geq (|G| - 1)m$. Notice that the rings $H^*(G)$ and $H^*(\mathcal{F}^n/G)$ coincide in dimensions $< n$ (see, for example, p. 356 of [5]); this makes it possible to carry out most of the computations within $H^*(G)$.

**Step 1.** If $G$ is not $q$-primary, then $e_q(\xi_\mu) = 0$.

Choose a $q$-Sylow subgroup $G_q$ of $G$. Further let $i: G_q \subseteq G$ and $\pi: \mathcal{F}^n/G_q \to \mathcal{F}^n/G$ be inclusion and projection, respectively. There is then a commutative diagram

\[
\begin{array}{cccccccc}
\ldots & \to & H^i(G) & \to & H^i(\mathcal{F}^n/G) & \to & H^{i-n}(G) & \to & H^{i+1}(G) & \to & \ldots \\
\downarrow i^* & & \downarrow \pi^* & & \downarrow i^* & & \downarrow i^* & & \\
\ldots & \to & H^i(G_q) & \to & H^i(\mathcal{F}^n/G_q) & \to & H^{i-n}(G_q) & \to & H^{i+1}(G_q) & \to & \ldots
\end{array}
\]

with the rows exact (see, for example, p. 358 of [5]) and $i^*$ monic (p. 259 of [5]). From the diagram one sees that

\[\pi^*: H^{[G]}(\mathcal{F}^n/G) \to H^{[G]}(\mathcal{F}^n/G_q)\]

is monic. Hence we just have to prove that $e_q(\pi^*\xi_\mu) = 0$.

By (30.14) of [9], RG$_q$-modules are isomorphic precisely when they have the same characters. An easy computation of characters then shows that

\[IG \simeq [G: G_q]IG_q \oplus ([G: G_q] - 1)R_{\text{triv}}\]

(as RG$_q$-modules). Here $R_{\text{triv}}$ is the the trivial RG$_q$-module. But then $\pi^*\xi_\mu$ splits off $([G: G_q] - 1)$ trivial line bundles and $e_q(\pi^*\xi_\mu) = 0$.

Now we concentrate on periodic $q$-primary groups; these are the cyclic groups $Z_{q^a}$ and (for $q = 2$) the generalized quaternion groups $Q_{2^a}$ of order $2^a \geq 8$ (see p. 262 of [5]).
STEP 2. If $G = Q^{2x}$ with $x \geq 3$, then $e_2(\xi_\mu) = 0$.

Recall that $Q^{2^x}$ has generators $a$, $b$ and relations $a^{2^{x-1}} = 1$, $b^2 = a^{2^{x-2}}$, $bab^{-1} = a^{-1}$. There are three non-trivial 1-dimensional $RQ^{2^x}$-modules $R_1, R_2, R_3$ given by their associated representations $T_i : Q^{2^x} \rightarrow Z_2$ as follows:

\[
T_1(a) = -1, \quad T_1(b) = 1, \\
T_2(a) = 1, \quad T_2(b) = -1, \\
T_3(a) = -1, \quad T_3(b) = -1.
\]

Clearly $R_3 = R_1 \otimes R_2$, and

\[
R_1 \oplus R_2 \oplus (R_1 \otimes R_2) = R_1 \oplus R_2 \oplus R_3
\]

is a submodule of $IQ^{2^x}$. With $\lambda_i$ denoting the real line bundle $(\mathcal{S}^n \times_{Q^{2^x}} R_i \rightarrow \mathcal{S}^n/Q^{2^x})$ it follows that $\lambda_1 \oplus \lambda_2 \oplus (\lambda_1 \otimes \lambda_2)$ is a subbundle of $\xi_\mu$. Hence $e_2(\xi_\mu)$ contains

\[
K = e_2(\lambda_1 \oplus \lambda_2 \oplus (\lambda_1 \otimes \lambda_2)) = e_2(\lambda_1)^2 e_2(\lambda_2) + e_2(\lambda_1) e_2(\lambda_2)^2
\]

as a factor.

In [13] it is shown that

\[
H^*(Q^{2^x}) = Z_2[x; y, w]/I_\alpha,
\]

where

\[
\deg x = \deg y = 1, \quad \deg w = 4, \\
I_3 = \{x^2 + xy + y^2, y^3\}, \\
I_\alpha = \{x^2 + xy, y^3\} \quad \text{for } \alpha > 3.
\]

But then (no matter what is $e_2(\lambda_i)$) one gets $K = 0$. And $e_2(\xi_\mu) = 0$ as promised.

STEP 3. If $G = Z_{q^x}$, then $e_q(\xi_\mu)^m = 0$ unless

\[
\alpha = 1, \quad n \geq (q - 1)m \quad \text{or} \quad q = 2, \quad \alpha = 2, \quad m = 1, n \geq 3.
\]

Make the complex numbers $C$ into a $CZ_{q^x}$-module by the formula

\[
e_g = \exp (2\pi (1)^{-1} q^{-s})
\]

($g$ a generator of $Z_{q^x}$). If $q = 2$, make the real line $R$ into an $RZ_{2^x}$-module using the formula $rg = -r$. If $\lambda$ is the principal $Z_{q^x}$-bundle $(\mathcal{S}^n \rightarrow \mathcal{S}^n/Z_{q^x})$, consider the complex [real] line bundle

\[
\lambda_C = (\mathcal{S}^n \times_{Z_{q^x}} C \rightarrow \mathcal{S}^n/Z_{q^x}), \\
[\lambda_R = (\mathcal{S}^n \times_{Z_{2^x}} R \rightarrow \mathcal{S}^n/Z_{2^x})]
\]
obtained by screwing in $C [R]$ as fiber in $\lambda$. Also for any complex vector bundle $\zeta$ denote its underlying real vector bundle by $\varrho(\zeta)$, and let $\xi^l$ be the complex tensor product of $l$ copies of $\zeta$.

A comparison of characters reveals the following isomorphisms of $RZ_{q^2}$-modules

\[
IN_{q^2} = C \oplus (C \otimes C) \oplus \ldots \oplus (C \otimes C \otimes \ldots \otimes C), \quad q \text{ odd},
\]
\[
IN_{2^2} = R \oplus C \oplus (C \otimes C) \oplus \ldots \oplus (C \otimes \ldots \otimes C), \quad q = 2,
\]
from which one gets immediately

\[(3.3) \quad \xi^l = \varrho(\lambda_C) \oplus \varrho(\lambda_C^2) \oplus \ldots \oplus \varrho(\lambda_C^{i(q^a-1)}), \quad q \text{ odd}, \]
\[(3.4) \quad \xi^l = \lambda_R \oplus \varrho(\lambda_C) \oplus \varrho(\lambda_C^2) \oplus \ldots \oplus \varrho(\lambda_C^{2a-1}), \quad q = 2. \]

Using well-known properties of Euler classes these in turn imply

\[(3.5) \quad e_q(\xi^l) = \left(\frac{1}{2}(q^a-1)! \right) e_q(\lambda_C)^{i(q^a-1)}, \quad q \text{ odd}, \]
\[(3.6) \quad e_2(\xi^l) = (2^{a-1}-1)! e_2(\lambda_C)^{2^{a-1}-1} e_2(\lambda_R), \quad q = 2. \]

Since

\[
H^{q^a-1}(\mathcal{S}^n/Z_{q^a}) = Z_q
\]

(if $n > q^a - 1$ this is a fact about $H^*(Z_{q^a})$; for $n = q^a - 1$ it is a fact about the ($Z_q$-orientable) manifold $\mathcal{S}^n/Z_{q^a}$), the numerical factors in (3.5) and (3.6) will kill $e_q(\xi^l)$ except for $q$ odd and $a = 1$ or $q = 2$ and $a \leq 2$.

Hence step 3 may be finished by showing that $e_2(\xi^l)^m = 0$ for $m > 1$, $q = 2$, and $a > 1$. But this is obvious since $e_2(\lambda_R)^2 = 0$ (recall that the one-dimensional generator of $H^*(Z_{2^a})$ has vanishing square when $a > 1$).

The above 3 steps prove the last part of the proposition. The first part will be proved in the next three steps.

\textbf{Step 4.} With $G = Z_q$ and $q$ odd, $e_q(\xi^l)^m \neq 0$ for all $m$ with $n \geq (q-1)m$.

Using Lefschetz fixed point theorem in the form given on p. 224 of [10], say, it is easily seen that $n$ must be odd. Hence $n$ is actually $(q-1)m$, and all computations can take place in $H^*(Z_q)$. Since

\[
e^q(\xi^l)^m = \left(\frac{1}{2}(q^a-1)! \right)^m e^q(\lambda_C)^{i(q^a-1)m}
\]

with

\[
e^q(\lambda_C) \in H^a(Z_q)
\]

and

\[
H^*(Z_q) = A(x) \otimes Z_q(\beta x)
\]

for any non-zero $x \in H^1(Z_q)$, it is sufficient to show that $e_q(\lambda_C)$ is non-zero.
To do so identify the set of isomorphism classes of principal $Z_q$-bundles over $\mathcal{S}^n/Z_q$ with

$$H^1(\mathcal{S}^n/Z_q) = H^1(Z_q)$$

(for this identification, see [11]). Then $\lambda$ becomes a generator of $H^1(Z_q)$ (a cross section of $\lambda$ would give rise to a $Z_q$-equivariant map $\mathcal{S}^n \to Z_q$ which is nonsense). Therefore $\beta \lambda \not\equiv 0$. But $\beta \lambda$ is precisely $e_q(\lambda_C)$ (this is most easily seen in the universal example $S^N \to S^n/Z_q$).

**Step 5.** With $G = Z_2$ one has $e_2(\xi_\mu)^m \equiv 0$ for all $m$ with $n \geq m$.

Since

$$e_2(\xi_\mu)^m = e_2(\lambda_R)^m$$

and

$$H^*(Z_2) = Z_2[x]$$

with $x$ one-dimensional, it suffices to show that $e_2(\lambda_R) \equiv 0$ (if $n = m$ use non-singularity of the cup-product-pairing to the top-dimension). But

$$e_2(\lambda_R) = w_1(\lambda_R) = \lambda$$

under the identification of $H^1(\mathcal{S}^n/Z_q)$ with the set of isomorphism classes of principal $Z_2$-bundles over $\mathcal{S}^n/Z_2$. Hence it is enough to see that $\lambda$ does not admit a section. This is obvious (as above in step 4).

**Step 6.** With $G = Z_4$ one has $e_2(\xi_\mu) \equiv 0$.

Here

$$e_2(\xi_\mu) = e_2(\lambda_R)e_2(\lambda_C)$$

and

$$H^*(Z_4) = \Lambda(x) \otimes Z_2[y]$$

with $x$ one-dimensional and $y$ two-dimensional. Hence one just has to prove that

$$e_2(\lambda_C) \equiv 0 \neq e_2(\lambda_R).$$

This is left to the reader.

4. The main theorem.

From propositions 2.1 and 3.1 one derives the following

**Theorem 4.1.** Let $\mu : \mathcal{S}^n \times Z_p \to \mathcal{S}^n$ be a proper action. Let $f : \mathcal{S}^n \to M^m$ be a nice map into an $m$-manifold $M^m$. If $p = 2$, assume that

$$f_* : H_n(\mathcal{S}^n ; Z_2) \to H_n(M^m ; Z_2).$$

Then

$$\text{cd}(A(\mu ; f)) \geq n - (p - 1)m.$$
Proof. Any $Z_p$-action on $S^n$ is automatically $Z_p$-orientation preserving.

The two theorems in the introduction are immediate corollaries (in view of the remarks after the definition of nice).

There is of course also a mod 4 Borsuk–Ulam theorem for nice maps $S^n \to S^1$.

5. Concluding remarks and an example.

I present first an example showing that the inequality in the mod $p$ Borsuk–Ulam theorem cannot in general be strengthened. View $S^{2n+1}$ as the unit sphere in $C^{n+1}$, and let $Z_p$ act upon $S^{2n+1}$ via multiplication by the powers of a $p$th root of unity $\rho$. Define

$$f = (f_1, \ldots, f_m) : S^{2n+1} \to R^m$$

by the formulae

$$f_i(z_0, \ldots, z_n) = \text{Im} \left( \sum_{v=1}^{k} \sigma_v(z_k(i-1), \ldots, z_{ki-1}) \right),$$

where $k = \frac{1}{2}(p-1)$, Im $z$ denotes the imaginary part of the complex number $z$, and $\sigma_v$ is the $v$th elementary symmetric polynomial. The defining condition for $A(\mu ; f)$ then reads

$$\text{Im} \left( \sum_{v=1}^{k} \sigma_v(z_k(i-1), \ldots, z_{ki-1})\rho_j^{v} - \sum_{v=1}^{k} \sigma_v(z_k(i-1), \ldots, z_{ki-1}) \right) = 0$$

for $j = 1, 2, \ldots, p$, $i = 1, 2, \ldots, m$. This system of equations may be solved as follows. Introduce the polynomials

$$Q_{z,i}(t) = \sum_{v=1}^{k} \sigma_v(z_k(i-1), \ldots, z_{ki-1})(t^v - 1).$$

The defining equations for $A(\mu ; f)$ then read

$$Q_{z,i}(\rho^j) \in R \quad \text{for } j = 1, 2, \ldots, p, \quad i = 1, 2, \ldots, m.$$

It is easy to prove that if $Q$ is a complex polynomial of degree $k = \frac{1}{2}(p-1)$ taking real values for all $p$th roots of unity, then $Q$ is a real constant. The solution of the defining equations for $A(\mu ; f)$ therefore is

$$\sigma_v(z_k(i-1), \ldots, z_{ki-1}) = 0, \quad v = 1, 2, \ldots, k, \quad i = 1, 2, \ldots, m;$$

that is,

$$A(\mu ; f) = \{(z_0, \ldots, z_n) \in S^{2n+1} \mid z_0 = \ldots = z_{km-1} = 0\}.$$

This is a sphere of dimension $2(n-km+1)-1 = 2n + 1 - (p-1)m$, hence of cohomological dimension $2n+1-(p-1)m$. 
Finally a few remarks concerning theorem 4.1: The most obvious $\mathbb{Z}_p$-action on a mod $p$ homology $n$-sphere is obtained as follows. Let $l$ be prime to $p$ and take the usual $\mathbb{Z}_p l$-action on $S^{2n+1}$. Factoring out the $\mathbb{Z}_l$-action gives an

$$S^{2n+1} = S^{2n+1}/\mathbb{Z}_l = L^{2n+1}_l$$

(a lens space) and there remains a proper $\mathbb{Z}_p$-action

$$\mu : L^{2n+1}_l \times \mathbb{Z}_p \to L^{2n+1}_l.$$ 

It seems unlikely that every map $f : L^{2n+1}_l \to M^m$ should be nice. However, it is easily seen that whether $f$ is nice or not, one has

$$\text{od}(A(\mu ; f), \mathbb{Z}_p) \geq (2n+1) - (p-1)m.$$ 

In fact, one just has to use the mod $p$ Borsuk–Ulam theorem on the composition

$$S^{2n+1} \xrightarrow{\pi} S^{2n+1}/\mathbb{Z}_l \xrightarrow{f} M^m$$

with $\mathbb{Z}_p$-action $\mu'$ on $S^{2n+1}$ and notice that

$$A(\mu', f\pi)/\mathbb{Z}_l = A(\mu ; f).$$

REFERENCES


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