ON $L^p$ ESTIMATES FOR
QUASI-ELLIPTIC BOUNDARY PROBLEMS

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The aim of this note is to indicate how the methods of [1] can be applied to derive a priori $L^p$-inequalities for quasi-elliptic boundary problems in the half-space $R_+^n$ when $1 < p < \infty$. However, we have comparatively stronger assumptions on the coefficients, as the norms are no longer rotation invariant.

1. Preliminaries.

Whenever convenient, we use the notations of [1]. If $\alpha$ is a multi-index, i.e. a sequence $\alpha_1, \ldots, \alpha_n$ of non-negative integers, we write

$$|\alpha| = \sum \alpha_j, \quad D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \text{ with } D_j = -i \partial/\partial x_j.$$ 

If $m$ is a sequence of positive integers $m_1, \ldots, m_n$, set

$$|\alpha:m| = \sum_{j=1}^m \alpha_j/m_j.$$ 

The constant coefficient differential operator

$$A = A(D) = \sum_{|\alpha:m| \leq 1} a_\alpha D^\alpha = A_0(D) + \sum_{|\alpha:m| < 1} a_\alpha D^\alpha$$

is “properly” quasi-elliptic, if the corresponding polynomial

$$A(\xi) = A(\xi_1, \xi'), \quad \xi' = (\xi_2, \ldots, \xi_n),$$

has the following properties:

(1) $A_0(\xi) \equiv 0$ for every $\xi \in R^n$, $\xi \neq 0$;

(2) for all real $\xi' \neq 0$, $A_0(\xi_1, \xi')$ has exactly $m_+$ zeros $\xi_1 = \psi(\xi')$ with positive and $m_- = m_1 - m_+$ zeros with negative imaginary part.

Because of the structure of the quasi-elliptic operator $A$, it is natural to introduce function spaces defined by use of the weight functions

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\[ A(\xi) = \delta \xi_1 + i \left( 1 + \delta^{2m_1} \sum_{j=2}^{n} \xi_j^{2m_j} \right)^{1/2m_1}, \quad \delta > 0. \]

Thus for \( 1 < p < \infty \) we take
\[ H_{s; p} = \{ f : f \in S', \, \bar{F}A^* Ff \in L^p \}, \]
where \( F \) is the Fourier transform, normed by
\[ \|f\|_{s; p} = \|\bar{F}A^* Ff\|_p. \]

Let \( (H_{s; p})_+ \) consist of those elements of \( H_{s; p} \), whose supports are contained in the half-space \( x_1 \leq 0 \), and set
\[ H_{s; p}^+ = H_{s; p}/(H_{s; p})_-, \]
with the quotient norm \( \|\cdot\|_{s; p}^+ \).

The traces of the \( H_{s; p} \)-functions on \( x_1 = 0 \) belong to the spaces \( W_{s; p} \) (for exact statements see [2, p. 95]). For
\[ sm_i/m_1 = s_i > [s_i], \quad i = 2, \ldots, n, \]
let \( W_{s; p} \) be the space of functions
\[ u = u(x'), \quad x' \in R^{n-1}, \]
with for all \( j \)
\[ D_j^ku \in L^p, \quad k \leq [s_j] \quad \text{and} \quad |D_j^{[s_j]}u|_{s-[s_j], j} < \infty, \]
where
\[ |v|_{0, j} = \left( \int_0^\infty t^{-p\theta} \|A_{j, t}v\|_p^p dt/t \right)^{1/p} \]
and
\[ A_{j, t}v = v(x_2, \ldots, x_j + t, \ldots, x_n) - v(x_2, \ldots, x_n). \]

\( W_{s; p} \) is a Banach space, normed by
\[ \langle u \rangle_{s; p} = \sum_j \left\{ \sum_{k=0}^{[s_j]} \delta^{km_1/m_j} \|D_j^ku\|_p + \delta^s |D_j^{[s_j]}u|_{s-[s_j], j} \right\}. \]

For \( s < 0 \), we set \( W_{s; p} = (W_{-s; p})' \).

The next class of functions takes \( W_{s; p} \) into \( W_{s; p} \) by multiplication. For \( s \geq 0 \) denote by \( K_{s; p} \) the set of functions \( \varphi \), for which
\[ D_j^k\varphi \in L_\infty, \quad k \leq [s_j], \]
s\( j \) as above, and such that
\[ \sum_j \left[ \sum_{k=0}^{[s_j]} \left( \int_0^\infty t^{-p(s_j-[s_j])} \|A_{j, t}D_j^k\varphi\|_\infty^p dt/t \right)^{1/p} \right] < \infty. \]

For \( s < 0 \), set \( K_{s; p} = K_{-s; p}' \). As in [1, Theorem 1.5], we have
Lemma 1. If \( f \in W_{s;1} \) and \( \varphi \in K_{s;1} \), then \( \varphi f \in W_{s;1} \) and
\[
\langle \varphi f \rangle_{s;1} \leq (K \| \varphi \|_{\infty} + o(1)) \langle f \rangle_{s;1} \quad \text{as } \delta \to +0.
\]
Here \( K \) is independent of \( \varphi \), but \( o(1) \) depends on \( \varphi \).

2. A priori estimates.

Let
\[
B_j = B_j(D) = \sum_{|x| = m \leq n_j/m_1} b_{jx} D^x = B_{0j}(D) + \sum_{|x| = m < n_j/m_1} b_{jx} D^x
\]
be differential operators with constant coefficients, and let the order of \( D_1 \) in \( B_j \) equal the order \( v_j \) of \( D_1 \) in \( B_{0j} \). Let \( F_{ij}(\xi') \) be the characteristic matrix corresponding to the boundary problem
\[
A_0 u = f \quad \text{in } R^+_n, \\
B_{0j} u = g_j \quad \text{on } x_1 = 0, j = 1, \ldots, m_+, 
\]
so that
\[
\sum_{j=0}^{m_+-1} F_{ij}(\xi') D_1 u(0, \xi') = B_{ij}(D_1, \xi') u(0,\xi'), \quad i = 1, \ldots, m_+, 
\]
if \( A u = 0 \) in \( R^+_n \). We require that
\[
\det F_{ij}(\xi') \neq 0 \quad \text{for } \xi' \neq 0. \tag{3}
\]
Then using Krée's version of Mihlin's theorem (cf. [3, Th. 8, p. 74]) and reasoning as in [1] we obtain

**Theorem 1.** Let \( A(D) \) and \( B_1(D), \ldots, B_{m_+}(D) \) satisfy (1), (2) and (3). For
\[
s m_j/m_1 \equiv 1/p \mod 1, \quad j = 1, \ldots, n,
\]
\[
s > s_0 = \max(m_+ - 1, v_1, \ldots, v_{m_+}) + 1/p,
\]
the following a priori estimate holds:
\[
\|u\|_{s;1}^+ \leq C \left( \delta^m \|Au\|_{s-m_1;1}^+ + \sum_{j=1}^{m_+} \delta^{n_j+1/p} \langle B_j u(0, \cdot) \rangle_{s-n_j-1/p;1} + \|u\|_{s-1;1}^+ \right).
\]
Here \( C \) is independent of \( u \in H_{s;1} \) and \( \delta \in (0, \delta_0) \) for some \( \delta_0 > 0 \).

Let finally
\[
A(x,D) = \sum_{|x| = m \leq 1} a_{a}(x) D^x
\]
be a differential operator with \( C^\infty \)-coefficients in \( R_+^n \), and let
be differential operators with $K_{s-n_j-1/p;p}$-coefficients on $x_1=0$, such that, for every $x^0 \in R^n_+$, $A(x^0,D)$ is properly quasi-elliptic and of order $m_j$ in $D_j$ and such that if $x_1^0=0$, the conditions of Theorem 1 are satisfied for the corresponding "frozen" operators $A(x^0,D)$ and $B_j(x^0,D)$. Then using Theorem 1 and Lemma 1, we immediately obtain

**Theorem 2.** Under the assumption of (4) the following a priori estimate holds:

$$
\|u\|_{s;p}^+ \leq C \left( \delta^{m_1} |A^u|_{s-m_1;p}^+ + \sum_{j=1}^{m_+} \delta^{n_j+1/p} \langle B_j^u(0, \cdot) \rangle_{s-n_j-1/p;p}^+ + |u|_{s-1;p}^+ \right)
$$

for $u \in H_{s;p}^+$ with compact support, the constant $C$ depending only on the support of $u$.

**References**


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