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SYMPLECTIC BUNDLES AND KR-THEORY

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In [4] Atiyah considered the category of spaces with involution and extended the definition of a real bundle to bundles over a space in this category. Using his notation, Atiyah shows (Corollary 3.8) that, for any $p \geqq 3$ there is a short exact sequence

$$(1) \qquad 0 \to KR^{-q}(X) \to KR^{-q}(X \times S^{p,0}) \to KR^{p+1-q}(X) \to 0\ .$$

Furthermore he shows (3.13) that this sequence splits. It is clearly sufficient to take $q=0$, and the inclusion $S^{3,0} \to S^{p,0}$, $p \geqq 3$, shows that it suffices to split the sequence

$$(2) \qquad 0 \to KR(X) \to KR(X \times S^{3,0}) \to KR^{4}(X) \to 0\ .$$

This he does by referring to some considerations about real algebraic manifolds. In this paper we shall establish this splitting by introducing symplectic bundles in the category. Therefore we shall deal with symplectic bundles on their own, and it is seen that they are quite as easy to handle as real bundles. For example, one gets the splitting principle for such bundles. Once and for all we refer to the above paper for notation.

A space X with involution τ we call an r-space.

We recall the notion of a real bundle in

Definition. *Let (X,τ) be an r-space and E a complex vector bundle over X and p: $E \to X$ the projection. Let τ: $E \to E$ be a continuous map such that $p\tau = \tau p$ and that for any $x \in X$ the map τ: $E_x \to E_{\tau x}$ is anti-linear.*

If $\tau^2 = id$, then (E,τ) is an r-bundle.
If $\tau^2 = -id$, then (E,τ) is an sp-bundle.

For an sp-bundle, τ: $E \to E$ is called an anti-involution. By the dimension of an r- or sp-bundle we mean the complex dimension. In particular an sp-line-bundle has complex dimension 1.

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EXAMPLES.

1) A quarternionic bundle over a space with trivial involution is an sp-bundle with $\tau =$ multiplication by $j \in \mathsf{H} = \mathsf{C} \oplus \mathsf{C}j$, the field of quarternions.

2) $X \times \mathsf{H}^n$ where $\mathsf{H}^n = \mathsf{C}^n \oplus \mathsf{C}^n j$ as complex vector space. The anti-involution is $(x,\underline{a}) \to (\tau x, j \cdot \underline{a})$.

3) Let $X = \{-1,1\}$ with involution $t \to -t$, $t \in \{-1,1\}$. The bundle $L_n = X \times \mathsf{C}^n$ has anti-involution $(t,\underline{a}) \to (-t, t \cdot \overline{\underline{a}})$, where $\overline{\underline{a}}$ is the complex conjugate of $\underline{a}$.

An sp-bundle is locally of the form 2) or 3).

$KR(X)$ and $Ksp(X)$ are defined as the Grothendieck groups of r- and sp-bundles, respectively. Put

$$KM(X) = KR(X) \oplus Ksp(X) .$$

We remark that $KM(X)$ is the Grothendieck group of complex bundles with a Z_4-action $\tau \colon E \to E$, satisfying $p\tau = \tau p$ and anti-linear on the fibres.

Under Whitney-sum and tensor-product, $KM(X)$ is a Z_2-graded ring with $KR(X)$ in degree 0 and $Ksp(X)$ in degree 1. Note also that the exterior power-operations λ^i induce operations $\lambda^i \colon KR \to KR$; $\lambda^{2j} \colon Ksp \to KR$ and $\lambda^{2j+1} \colon Ksp \to Ksp$.

Now most of §§ 2 and 3 of [4] can be worked out with KM instead of KR throughout. A detailed exposition is left to the reader. Especially we have:

Let $KR^{-8}(*) = Z(\lambda)$. Then multiplication by λ induces an isomorphism

$$KM^i(X) \xrightarrow[\lambda \cdot]{} KM^{i-8}(X) , \tag{3}$$

and

$$0 \to KM(X) \xrightarrow{\pi^*} KM(X \times S^{3,0}) \xrightarrow{\delta} KM^4(X) \to 0 . \tag{4}$$

is a short exact sequence.

Let E be an r- or sp-bundle over the r-space X and let $P(E)$ denote the complex projective space-bundle over X corresponding to E, with involution induced by $\tau \colon E \to E$. The Hopf-bundle $H = (H^*)^*$ is an r- or sp-line-bundle over $P(E)$ depending on the r- or sp-structure on E.

Especially consider the 2-dimensional trivial sp-bundle H over a point: $P(\mathsf{H})$ is easily seen to be equivalent to $S^{3,0}$ in the category of r-spaces. The Hopf-bundle over $P(\mathsf{H})$ is denoted H_{sp}. When we forget the sp-

structure, it is the usual complex Hopf-bundle over S^2. From (4) we get

$$0 \to Ksp(*) \to Ksp(S^{3,0}) \xrightarrow{\delta} Ksp^4(*) \to 0\,, \tag{5}$$

and $d_0 = \delta H_{sp}$ is the generator of $Ksp^4(*) = \mathsf{Z}$. It follows that $d_0{}^2 \in KR^8(*)$ is a generator and by means of (3) we get (compare Bott [6]):

Multiplication by d_0 induces isomorphisms

$$KR(X) \xrightarrow[d_0\cdot]{} Ksp^4(X), \qquad Ksp(X) \xrightarrow[d_0\cdot]{} KR^4(X)\,. \tag{6}$$

Now (4) gives the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & KR(X) & \to & KR(X \times S^{3,0}) & \to & KR^4(X) & \to 0 \\ & \downarrow{\scriptstyle d_0\cdot} & & \downarrow{\scriptstyle H_{sp}\cdot} & & & \\ 0 \leftarrow & Ksp^4(X) & \leftarrow & Ksp(X \times S^{3,0}) & \leftarrow & Ksp(X) & \leftarrow 0\,. \end{array} \tag{7}$$

Here the horizontal sequences are exact and the vertical maps are isomorphisms. Both sequences therefore split (compare [4]). In fact, we get

THEOREM 1.

$$\begin{aligned} KR(X \times S^{3,0}) &= KR(X)\cdot 1 \oplus Ksp(X)\cdot H_{sp}\,, \\ Ksp(X \times S^{3,0}) &= Ksp(X)\cdot 1 \oplus KR(X)\cdot H_{sp}\,. \end{aligned}$$

Note that $H_{sp} = \mathsf{H} - H_{sp}$ in $Ksp(S^{3,0})$.

Again following Atiyah [4], the exact sequences for the pairs $(X \times S^{2,0}, X \times S^{1,0})$ and $(X \times S^{3,0}, X \times S^{2,0})$ give:

For every r-space X there are exact sequences

$$\ldots \to K^{i-1}(X) \to KSC^i(X) \to K^i(X) \to K^i(X) \to \ldots\,, \tag{8}$$

$$\ldots \to K^i(X) \to KM^i(X) \to KSC^i(X) \to K^{i+1}(X) \to \ldots\,. \tag{9}$$

These sequences are due to Anderson [2]. Note that Theorem 1 is used in (9). It is not difficult in this set-up to determine the maps in (8) and (9). For example, the map $KM^i \to KSC^i$ sends a r- or sp-bundle, considered as a self-conjugate bundle, into itself. In the sequel we only use the following fact, which is clear from the construction of (8) and (9):

All maps in (8) *are $KSC^*(X)$-module homomorphisms and all maps in* (9) *are $KM^*(X)$-module homomorphisms.*

We now generalize our Theorem 1 and Theorem 2.1 in Atiyah [4].

THEOREM 2. *Let E be an r- or sp-bundle over the r-space X and let H be the Hopf-bundle over $P(E)$. Then $KM^*(P(E))$ is a free $KM^*(X)$-module with generators $1, H, \ldots, H^{n-1}$ ($n = \dim E$), and H satisfies the single relation*

$$\sum_{i=0}^{n} (-1)^i \lambda^i(E) H^i = 0 \quad in \quad KR(P(E)) .$$

PROOF. This theorem is proved in Atiyah [3] for $K^*(P(E))$. The first part of the theorem is therefore proved by means of the sequences (8) and (9) and the Five lemma.

Theorem 2 gives rise both to the Thom isomorphism for r-bundles and the splitting principle for KM in the category of spaces with involution. In fact, r- or sp-bundles split into sums of r- or sp-line-bundles, respectively, by pulling back to flag manifolds with involution. Therefore it is easy to handle operations. For example, just as in Atiyah [3], one defines Adams operations $\psi^k\colon KM \to KM$. These have all the usual properties, and in fact, when X has trivial involution, $\psi^k\colon KR(X) \to KR(X)$ are the original Adams operations. Note that $\psi^{2k}\colon Ksp \to KR$ and $\psi^{2k+1}\colon Ksp \to Ksp$.

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