SUMS AND INTERSECTIONS OF LEBESGUE SPACES

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To begin with, consider the following problem in the calculus of variations:

Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be \( n \) numbers in the open interval \((0, 1)\). We want to minimize the expression

\[
N = \sum_{i=1}^{n} \left[ \int_{-\infty}^{\infty} h_i(x) \, dx \right]^\alpha_i,
\]

where \( h_1, h_2, \ldots, h_n \) vary through all those non-negative functions in \( L^1 (\mathbb{R}) \) such that the sum

\[
g = \sum_{i=1}^{n} [h_i(x)]^{\alpha_i}
\]

remains fixed.

In this paper we shall prove, among other things, that this variational problem has a solution, that is, the infimum of \( N \) is actually attained. To fix our idea, we restrict ourselves to the case where \( n=2 \). The discussions of the general case will be completely parallel.

We use [3] and [2] as our chief references in real analysis and harmonic analysis respectively.

Let us change our notation to write \( \alpha_i = 1/q_i \) and \([h_i(x)]^{\alpha_i} = g_i(x)\), \( i=1,2 \). Then \( g_i \in L^{q_i} \), and (2) becomes

\[
g = g_1 + g_2.
\]

It is quite clear that the infimum of \( g \) is not changed if we allow \( g_i \) to take complex values so long as we insert absolute value signs under the signs of integration in (1):

\[
N = \|g_1\|_{q_1} + \|g_2\|_{q_2}.
\]

This suggests that we introduce the sum of the Lebesgue spaces \( L^{q_1} \) and \( L^{q_2} \), that is, the set of all functions \( g \) which are expressible in the form (3). In fact we have

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Theorem 1. Let \( S = S_{q_1,q_2} \) be the set of all complex-valued functions \( g \) of the form (3) with \( g_i \in L^{q_i} \), where \( 1 \leq q_i \leq \infty \), \( i = 1, 2 \). Then \( S \) is a Banach space if we supply it with the norm
\[
\|g\| = \inf N,
\]
where \( N \) is defined by (4) and the infimum is taken over all decompositions of \( g \) in the form (3).

Proof. There is no difficulty in verifying that \( S \) is a complex linear space and that (5) defines a semi-norm on it. To prove that (5) indeed gives a norm, assume that \( \|g\| = 0 \). Then there exist sequences \( g_1(n) \) in \( L^{q_1} \) and \( g_2(n) \) in \( L^{q_2} \) such that
\[
g = g_1(n) + g_2(n),
\]
and
\[
\lim \|g_1(n)\|_{q_1} = \lim \|g_2(n)\|_{q_2} = 0.
\]
As the last fact implies that both \( g_1(n) \) and \( g_2(n) \) converge to 0 in measure, \( g = 0 \) a.e. Hence (5) defines a norm.

It remains to be proved that \( S \) is complete under this norm. Thus let \( g(n) \in S \) be such that \( \sum_{n=1}^{\infty} \|g(n)\| < \infty \). Select \( g_1(n) \in L^{q_1} \) and \( g_2(n) \in L^{q_2} \) such that \( g(n) = g_1(n) + g_2(n) \) and
\[
\|g_1(n)\|_{q_1} + \|g_2(n)\|_{q_2} \leq \|g(n)\| + 2^{-n}.
\]
It follows from (6) that both \( \sum_{n=1}^{\infty} \|g_1(n)\|_{q_1} \) and \( \sum_{n=1}^{\infty} \|g_2(n)\|_{q_2} \) converge. Since \( L^{q_1} \) and \( L^{q_2} \) are complete, \( \sum_{n=1}^{\infty} g_1(n) \) and \( \sum_{n=1}^{\infty} g_2(n) \) exist in \( L^{q_1} \) and \( L^{q_2} \) respectively. Denote their sums by \( g_1 \) and \( g_2 \) respectively, and set \( g = g_1 + g_2 \). Then
\[
\|g - \sum_{n=1}^{k} g(n)\| \leq \|g_1 - \sum_{n=1}^{k} g_1(n)\|_{q_1} + \|g_2 - \sum_{n=1}^{k} g_2(n)\|_{q_2}.
\]
Hence \( \sum_{n=1}^{\infty} g(n) = g \) in \( S \). This completes the proof.

One consequence of this theorem is that the variational problem posed at the outset would be solved if we can show that the norm (5) of \( S \) is attained by a certain decomposition of \( g \) in the form (3) for \( 1 < q_i < \infty \), \( i = 1, 2 \). This will be done in Corollary 1.

Next we like to identify the dual space of \( S \). This will be done in the case where neither \( q_1 \) nor \( q_2 \) is \( \infty \). Let \( p_1 \) and \( p_2 \) be defined by
\[
\frac{1}{p_i} + \frac{1}{q_i} = 1, \quad i = 1, 2,
\]
with the usual convention of arithmetic on the symbol \( \infty \). Then for
any bounded linear functional $T$ on $S$, the restrictions of $T$ to $L^{q_i}$ are bounded linear functionals on $L^{q_i}$. Hence there exist functions $f_i \in L^{p_i}$ such that

$$Tg_i = \int_{-\infty}^{\infty} g_i(x) f_i(x) \, dx$$

for all $g_i \in L^{q_i}$. In particular, if $g \in L^{q_1} \cap L^{q_2}$, we have

$$Tg = \int_{-\infty}^{\infty} g(x) f_1(x) \, dx = \int_{-\infty}^{\infty} g(x) f_2(x) \, dx.$$  

Since $L^{q_1} \cap L^{q_2}$ includes the characteristic functions of all sets with finite measure, this implies that $f_1 = f_2$ a.e. Call their common value $f$. Then $f \in D$ where

$$D = D_{p_1, p_2} = L^{p_1} \cap L^{p_2}.$$

If $g = g_1 + g_2$ is a decomposition of $g$ in the form (3), then

$$Tg = \int_{-\infty}^{\infty} g_1(x) f(x) \, dx + \int_{-\infty}^{\infty} g_2(x) f(x) \, dx,$$

that is,

$$Tg = \int_{-\infty}^{\infty} g(x) f(x) \, dx.$$  

(8) 

Conversely, it is quite evident that, for any $f \in D$, (8) defines a bounded linear functional on $S$.

Next we would like to calculate $\|T\|$ where $T$ is defined by (8). Decompose $g$ according to (3) as $g = g_1 + g_2$. Then

$$|Tg| \leq \|g_1\|_{q_1} \|f\|_{p_1} + \|g_2\|_{q_2} \|f\|_{p_2} \leq N \max(\|f\|_{p_1}, \|f\|_{p_2}).$$

Hence

$$|Tg| \leq \|g\| \max(\|f\|_{p_1}, \|f\|_{p_2}).$$

Hence $\|T\| \leq \max(\|f\|_{p_1}, \|f\|_{p_2})$. We shall prove that equality actually holds here. This is trivially true if $f = 0$ a.e. Otherwise assume without loss of generality that $\|f\|_{p_1} \geq \|f\|_{p_2}$. Let a number $\varepsilon$ in $(0, \|f\|_{p_1})$ be given. Then there exists a function $g \in L^{q_1} \subset S$, not a.e. 0, such that

$$\left| \int_{-\infty}^{\infty} g(x) f(x) \, dx \right| \geq \|g\|_{q_1}(\|f\|_{p_1} - \varepsilon).$$
Since $g = g_1 + 0$ is a decomposition of $g$ of the type (3), it follows from (8) and (5) that

$$||T||g|| \geq |Tg| \geq (||f||_{p_1} - \varepsilon)||g||_{q_1} \geq (||f||_{p_1} - \varepsilon)||g||.$$

Hence

$$||T|| \geq ||f||_{p_1} - \varepsilon = \max(||f||_{p_1}, ||f||_{p_2}) - \varepsilon.$$

Since $\varepsilon$ is arbitrary, it follows that

$$||T|| = \max(||f||_{p_1}, ||f||_{p_2}).$$

Thus we have proved the following two theorems:

**Theorem 2.** Let $p_1$ and $p_2$ be two numbers in $(1, \infty]$ and let $D = D_{p_1,p_2} = L^{p_1} \cap L^{p_2}$. Then $D$ is a Banach space if we supply it with the norm

$$||f|| = \max(||f||_{p_1}, ||f||_{p_2}), \quad f \in D.$$

**Theorem 3.** If $q_1$ and $q_2$ are numbers in $[1, \infty)$, then the conjugate space of $S = S_{q_1,q_2}$ is isometrically isomorphic to $D = D_{p_1,p_2}$, where $p_i$ and $q_i$ are related by (7) and the operation of $f \in D$ on $g \in S$ is given by (8).

Actually, Theorem 2 remains valid even if we allow $p_1$ and $p_2$ to vary in $[1, \infty]$. The proof of this fact is similar to (and simpler than) that of Theorem 1, and, accordingly, will be omitted.

Our next task is to find the conjugate space of $D$ when $p_1$ and $p_2$ are in $[1, \infty)$. This is given by

**Theorem 4.** If $p_1$ and $p_2$ are numbers in $[1, \infty)$, then the conjugate space of $D = D_{p_1,p_2}$ is isometrically isomorphic to $S = S_{q_1,q_2}$, where $p_i$ and $q_i$ are related by (7) and the operation of $g \in S$ on $f \in D$ is given by

$$T(f) = \int_{-\infty}^{\infty} f(x) g(x) \, dx.$$

**Proof.** Clearly for every $g \in S$, the functional $T$ defined by (9) is linear. Further, if $g = g_1 + g_2$, $g_i \in L^{q_i}$, is a decomposition of $g$ in the form (3), then

$$|Tf| \leq \left| \int_{-\infty}^{\infty} f(x) g_1(x) \, dx \right| + \int_{-\infty}^{\infty} f(x) g_2(x) \, dx \leq ||f||_{p_1} ||g_1||_{q_1} + ||f||_{p_2} ||g_2||_{q_2} \leq ||f|| (||g_1||_{q_1} + ||g_2||_{q_2}).$$
But this implies that $T$ is bounded and $\|T\| \leq \|g\|$. 

Because $D$ contains the characteristic functions of all sets with finite measures, it follows from (9) that the correspondence $g \mapsto T$ is one-to-one. We have still to show that this correspondence is onto and that $\|T\| = \|g\|$. 

To do so, consider the Banach space $L^{p_1} \oplus L^{p_2}$ with the norm 

$$\|(f_1, f_2)\| = \max(\|f_1\|_{p_1}, \|f_2\|_{p_2}).$$ 

Then $D$ can be embedded in this space as its diagonal, that is, for $f \in D$, define $\varphi(f) = (f, f)$. Then $\varphi$ maps $D$ isometrically into $L^{p_1} \oplus L^{p_2}$. Now let $T$ be a bounded linear functional on $D$. Then, by the Hahn–Banach theorem, $T \circ \varphi^{-1}$ has a norm preserving extension to $L^{p_1} \oplus L^{p_2}$. As the conjugate space of $L^{p_1} \oplus L^{p_2}$ is clearly $L^{q_1} \oplus L^{q_2}$ with the norm 

$$\|(g_1, g_2)\| = \|g_1\|_{q_1} + \|g_2\|_{q_2}, \quad g_i \in L^{q_i}, \ i = 1, 2,$$

there are functions $g_1 \in L^{q_1}$ and $g_2 \in L^{q_2}$ such that 

$$T(f) = \int_{-\infty}^{\infty} f(x) g_1(x) \, dx + \int_{-\infty}^{\infty} f(x) g_2(x) \, dx.$$

Define $g = g_1 + g_2$. Then $g \in S$ and $T(f)$ is given by (9) for this $g$. As the norm of $T$ is the same as its extension, we have 

(10) 

$$\|T\| = \|g_1\|_{q_1} + \|g_2\|_{q_2} \geq \|g\|.$$

But we proved before that $\|T\| \leq \|g\|$. Hence we get $\|T\| = \|g\|$, and the theorem is proved.

It follows from (10) that 

(11) 

$$\|g\| = \|T\| = \|g_1\|_{q_1} + \|g_2\|_{q_2}.$$

Hence, 

**Corollary 1.** For each $g \in S$, there exist functions $g_1 \in L^{q_1}$ and $g_2 \in L^{q_2}$ such that $g = g_1 + g_2$ and that 

$$\|g\| = \|g_1\|_{q_1} + \|g_2\|_{q_2}.$$

This Corollary implies that the variational problem stated at the beginning of this paper always has a solution, as we have mentioned before.

**Corollary 2.** If $p_1$ and $p_2$ are in $(1, \infty)$, then $D=D_{p_1, p_2}$ is reflexive. If $q_1$ and $q_2$ are in $(1, \infty)$, then $S=S_{q_1, q_2}$ is reflexive.
Although Theorem 3 does not cover the case where $D=D_{1,p}$ with $p$ in $(1,\infty]$, it is still true that $D$ is a conjugate space in this case. More precisely, we have

**Theorem 5.** Let $q$ be a number in $[1,\infty)$ and let $C_0$ denote the set of all continuous functions on the real line which vanish at infinity. Denote by $\Sigma=\Sigma_q$ the set of all functions on the real line which can be written as

$$g = g_1 + g_2,$$

where $g_1 \in C_0$ and $g_2 = L^q$. For $g \in \Sigma$, define

$$||g|| = \inf(\sup|g_1| + ||g_2||_q),$$

where the infimum is taken over all decompositions of $g$ in the form (12). Then $\Sigma$ becomes a Banach space with respect to this norm. Further, the conjugate space of $\Sigma$ is isometrically isomorphic to $D=D_{1,p}$ with

$$\frac{1}{p} + \frac{1}{q} = 1,$$

the operation of $f \in D$ on $g \in \Sigma$ being given by (8).

**Proof.** That $\Sigma$ is a Banach space is proved by the same argument, *mutatis mutandis*, as that for Theorem 1. That the conjugate space of $\Sigma$ is isometrically isomorphic to $D$ is proved by the same argument, likewise *mutatis mutandis*, as that for Theorem 3. The only non-trivial modification made here occurs when we want to show that every bounded linear functional $T$ on $\Sigma$ is given by (8) for some function $f \in D$. This is done in the following manner: Since $C_0 \subset \Sigma$, the restriction of $T$ on $C_0$ defines a bounded linear functional on $C_0$. Hence there is a complex bounded Radon measure $\nu$ on $(-\infty,\infty)$ such that

$$Tg = \int_{-\infty}^{\infty} g(x) \, d\nu(x), \quad g \in C_0.$$  

Similarly, since $L^q \subset \Sigma$, there is a function $f \in L^p$ such that

$$Tg = \int_{-\infty}^{\infty} g(x) f(x) \, dx, \quad g \in L^q.$$  

In particular, if $g$ is a continuous function with compact support, then both formulas are valid, and
\[ \int_{-\infty}^{\infty} g(x) \, dv(x) = \int_{-\infty}^{\infty} g(x) \, f(x) \, dx. \]

This implies that \( v \) is absolutely continuous and
\[ dv(x) = f(x) \, dx. \]

Therefore, \( f \in L^1 \). Hence \( f \in D \). Now if \( g \) is an arbitrary function in \( \Sigma \), let \( g = g_1 + g_2 \) be a decomposition of \( g \) in the form of (12). Then
\[ Tg = Tg_1 + Tg_2 = \int_{-\infty}^{\infty} g_1(x) \, dv(x) + \int_{-\infty}^{\infty} g_2(x) \, f(x) \, dx \]
\[ = \int_{-\infty}^{\infty} g(x) \, f(x) \, dx, \]
which is (8). This proves the theorem.

There are a few nooks and corners which should be cleared up before we go on. First, if \( 1 \leq q_1 < q_2 < q_3 \leq \infty \), then any function \( g_2 \in L^{q_2} \) can be represented as \( g_1 + g_3 \) with \( g_1 \in L^{q_1} \) and \( g_3 \in L^{q_3} \). In fact, one way to define \( g_1 \) and \( g_3 \) is
\[ g_1(x) = \begin{cases} g_2(x) & \text{if } |g_2(x)| > 1, \\ 0 & \text{if } |g_2(x)| \leq 1, \end{cases} \quad g_3 = g_2 - g_1. \]
Hence if we define \( S = L^{q_1} + L^{q_2} + L^{q_3} \) with the norm
\[ \|g\| = \inf \left( \|g_1\|_{q_1} + \|g_2\|_{q_2} + \|g_3\|_{q_3} \right), \]
where the infimum is taken over all decompositions \( g = g_1 + g_2 + g_3, \ g_1 \in L^{q_1} \), then \( S = S_{q_1,q_3} \), as sets, and a simple application of the open mapping theorem implies that this norm of \( g \) in \( S \) is equivalent to its norm in \( S_{q_1,q_3} \). A similar reasoning applies to the space \( D = L^{p_1} \cap L^{p_2} \cap L^{p_3} \). Further, this generalizes to any number of indices \( p_i \) and \( q_i \). The variational problem for (1) when \( n > 2 \) is solved, however, by the previous technique which we used for \( n = 2 \).

Secondly, the solution of our variational problem is in general not unique. Thus, let \( g \) be the characteristic function of the unit interval \([0,1]\). Then for any \( g \) in \((1,\infty), \ \|g\|_{q_2} = 1 \). If \( g = g_1 + g_2 \) where \( g_1 \in L^{q_1}, \ g_2 \in L^{q_2}, \ q_1 < q_2, \) and both \( g_1 \) and \( g_2 \) are supported in \([0,1]\), then \( \|g_2\|_{q_2} \geq \|g_2\|_{q_1} \). Hence
\[ \|g_1\|_{q_1} + \|g_2\|_{q_2} \geq \|g_1\|_{q_1} + \|g_2\|_{q_1} \geq \|g\|_{q_1} = 1. \]
Hence
\[ \|g\| = 1 = \|g\|_{q_1} + \|0\|_{q_2} = \|\frac{1}{2}g\|_{q_1} + \|\frac{1}{2}g\|_{q_2}. \]
Thus, both \((g_1, g_2) = (g, 0)\) and \((g_1, g_2) = (\frac{1}{2}g, \frac{1}{2}g)\) play the roles of solutions of the variational problem.

Thirdly, for \(g \in S\) we define
\[ \|\|g\||\| = \inf \max (\|g_1\|_{q_1}, \|g_2\|_{q_2}), \]
where the infimum is taken over all decompositions \(g = g_1 + g_2\) with \(g_i \in L^{q_i}, i = 1, 2;\) and for \(f \in D\), we define
\[ \|\|f\||\| = \|f\|_{p_1} + \|f\|_{p_2}. \]
Then (13) and (14) provide norms equivalent to the original norms on \(S\) and \(D\) respectively. Further, Theorems 3 and 4 remain valid with these norms. Other equivalent norms are also feasible, and to each of these equivalent norms on \(S\) there corresponds a solvable variational problem.

Fourthly, the assumption that the underlying measure in our definitions of \(S\) and \(D\) is the Lebesgue measure on \((-\infty, \infty)\) is made only for the simplicity of exposition. In fact, we can use any measure space \((X, \mathcal{B}, \mu)\) in our definitions of \(S\) and \(D\). Theorems 1 and 2 remain valid in this general setting. Theorem 3 is true in general if neither \(q_1\) nor \(q_2\) is 1, and is true even for these indices if \(\mu\) is \(\sigma\)-finite. Similarly, Theorem 4 is true in general if neither \(p_1\) nor \(p_2\) is 1, and is true even for these indices if \(\mu\) is \(\sigma\)-finite. Theorem 5 does not make sense unless there is some sort of topology on \(X\), and it becomes a valid theorem if \(X\) is a locally compact space and \(\mu\) is a Radon measure on it.

From now on we shall consider a locally compact group \(G\) and its left Haar measure \(\mu\). In this case the set \(D = D_{1, p},\ p > 1,\) has some additional structure:

**Lemma.** \(D = D_{1, p} = L^1(G, \mu) \cap L^p(G, \mu)\) is a dense left ideal in \(L^1\), where \(L^1 = L^1(G, \mu)\) is considered as a Banach algebra with convolution
\[ f \ast g(x) = \int_G f(y) g(y^{-1}x) \, d\mu(y) \]
as multiplication.

**Proof.** \(D\) is a left ideal because \(L^1\)-functions operate boundedly linearly by convolutions from the left. \(D\) is dense in \(L^1\) because it contains all the continuous functions on \(G\) with compact supports.
Theorem 6. D with its own norm is a Banach algebra under convolution. Further, D is commutative if and only if G is abelian.

Proof. Since D is known to be a Banach space and an ideal in L¹, the first statement will follow if we show that \( \|f \ast g\| \leq \|f\| \cdot \|g\| \) for all \( f, g \in D \). Indeed,

\[
\|f \ast g\| = \max(\|f \ast g\|_1, \|f \ast g\|_p) \leq \|f\|_1 \cdot \|g\| \leq \|f\| \cdot \|g\|.
\]

Hence we have the first statement. The second statement follows from a similar statement for \( L^1 \) and the density of D in \( L^1 \).

We are going to study the ideal theory in \( D = D_{1,p} \) where \( 1 < p < \infty \). For a closed right ideal \( I_1 \subset L^1 \) define

(15) \[ \delta(I_1) = I_1 \cap D. \]

Then clearly \( \delta(I_1) \) is a closed right ideal in D.

Theorem 7. By (15) is defined a one–one mapping \( \delta \) from the set of all closed right ideals in \( L^1 \) onto the set of all closed right ideals in \( D = D_{1,p} \), \( 1 < p < \infty \). Further, \( \delta(I_1) \) is a two-sided ideal in D if and only if \( I_1 \) is a two-sided ideal in \( L^1 \).

Proof. It is known that there is a net \( u_\alpha \) of continuous functions with compact supports such that

\[ \|f - f \ast u_\alpha\|_r \to 0 \quad \text{for each } f \in L^r, \quad 1 \leq r < \infty. \]

We shall prove our theorem by dint of this net.

First, let \( \delta(I_1) = I \) and let \( J_1 \) be the \( L^1 \)-closure of \( I \). Clearly \( J_1 \subset I_1 \). On the other hand, if \( f \in I_1 \), then each \( f \ast u_\alpha \in I_1 \), since \( u_\alpha \in L^1 \) and \( I_1 \) is a right ideal in \( L^1 \). Also \( f \ast u_\alpha \in D \) since \( u_\alpha \in D \) and D is a left ideal in \( L^1 \). Thus \( f \ast u_\alpha \in I \). As \( f \ast u_\alpha \to f \) in \( L^1 \), we get \( f \in J_1 \). This proves that \( I_1 = J_1 \). In particular, \( \delta \) is a one–one mapping.

Next let \( I \) be a closed right ideal in \( D \) and let \( I_1 \) be the \( L^1 \)-closure of \( I \). We want to show that \( I = \delta(I_1) \). For this, take \( f \in \delta(I_1) \). Then there is a sequence \( f_n \in I \) such that \( f_n \to f \) in \( L^1 \). Since convolution is continuous,

\[ f_n \ast u_\alpha \to f \ast u_\alpha \quad \text{as} \quad n \to \infty \]

both in \( L^1 \) and in \( L^p \), hence in \( D \). As \( I \) is a right ideal and as \( u_\alpha \in D \), \( f_n \ast u_\alpha \in I \). Hence \( f \ast u_\alpha \in I \) for each \( \alpha \), since \( I \) is closed. Finally \( f \ast u_\alpha \to f \) both in \( L^1 \) and in \( L^p \), hence in \( D \). This yields that \( f \in I \). Hence \( I = \delta(I_1) \). This means that the mapping \( \delta \) is onto.
It is quite clear that if \( I_1 \) is two-sided, then \( \delta(I_1) \) is also two-sided. Conversely, if \( \delta(I_1) \) is two-sided, then \( I_1 \), being its \( L^1 \)-closure, is also two-sided, by the density of \( D \) and the joint continuity of convolution in \( L^1 \).

One consequence of Theorem 7 is that the maximal ideals of \( L^1 \) and those of \( D \) correspond to each other. It says nothing, however, of the correspondence between their regular maximal ideals. We prove now that this holds for any abelian group \( G \).

**Theorem 8.** Suppose \( G \) is abelian; then for each \( p \) in \((1, \infty] \), the maximal ideal space of \( D = D_{1,p} \) is homeomorphic to the dual group \( \hat{G} \) of \( G \).

**Proof.** Let \( f \) be a non-zero element of \( D \). Then for each integer \( n > 1 \) we have

\[
\|f^n\| = \max(\|f^{n-1} \ast f\|_1, \|f^{n-1} \ast f\|_p) \leq \|f\|_1^{n-1} \|f\|.
\]

Extracting \( n \)th roots on both sides and then letting \( n \to \infty \), we get

\[
e(f) \leq \|f\|_1,
\]

where \( e(f) \) denotes the spectral radius of \( f \). Clearly (16) also holds for \( f = 0 \).

Now let \( F \) be a multiplicative linear functional on \( D \). Then \( |F(f)| \leq e(f) \leq \|f\|_1 \) for each \( f \in D \). Since \( D \) is dense in \( L^1 \), \( F \) has a unique extension to a multiplicative linear functional on \( L^1 \). As the maximal ideal space of \( L^1 \) can be identified to \( \hat{G} \), there exists an element \( \xi \in \hat{G} \) such that

\[
F(f) = \int_{\hat{G}} (\xi, \xi) f(x) d\mu(x)
\]

for each \( f \in D \). Conversely, if \( \xi \in \hat{G} \), the functional \( F \) on \( D \) defined by (17) is multiplicative and linear. Also, different \( \xi \)'s determine different \( F \)'s because \( D \) is dense in \( L^1 \).

Another consequence of the density of \( D \) in \( L^1 \) is that the Gelfand topologies of \( \hat{G} \) as maximal ideal spaces of \( D \) and of \( L^1 \) coincide. Theorem 8 is therewith proved.

Combining Theorems 7 and 8, we see that if \( G \) is abelian and if \( 1 < p < \infty \), then Wiener's Tauberian theorem holds for \( D = D_{1,p} \). Also, spectral synthesis fails for \( D_{1,p} \), where \( 1 < p < \infty \), unless \( G \) is discrete, which is a trivial case because then \( D_{1,p} = L^1 \).
If $G$ is compact, then $D = L^p$. Hence Theorems 7 and 8 hold for the $L^p$ algebras of compact groups.

Theorems 6, 7 and 8 are suggested by some similar results in a previous paper [1].

**Added in Proof.** The editor brought to our attention the paper *The structure space of a left ideal*, Math. Scand. 14 (1964), 90–92, by G. K. Pedersen, where it is proved that if $D$ is a left ideal in a ring $L$, then to each maximal regular right ideal $I$ in $D$ there corresponds a maximal regular right ideal $J$ in $L$ such that $J \cap D = I$. Furthermore there is a homeomorphism between the right structure space of $D$ and the (open) set of right primitive ideals in $L$ not containing $D$.

Now in our situation where $L = L^1$ is an involutive algebra and $D$ is a dense left ideal, this yields that the structure space of $D$ is homeomorphic to the structure space of $L$, the latter being the kernels in $L$ of irreducible unitary representations of the group $G$. When $G$ is abelian, this result combined with Theorem 8 gives that the Gelfand topology coincides with the Jacobson (hull-kernel) topology on the maximal ideal space of $D$.

**References**


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