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ON THE STRUCTURE OF MAXIMALLY ALMOST PERIODIC GROUPS

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Theorems 1 and 2 of section 2 indicate the importance of "finitely orbited representations of a normal subgroup" and the existence of closed (open) subgroups of finite index in maximally almost periodic (MAP) groups. In theorem 4 the ideas of theorems 1 and 2 are combined to obtain a characterization of MAP semidirect products dependent only on the representations of the factors. In section 3 some results concerning the structure of MAP groups with normal Abelian subgroups are presented. In particular, MAP groups with non-central normal Abelian subgroups have "many" subgroups of finite index. See theorem 6. Theorems 1, 2, and 4 in this work were previously announced in [14].

1. Definitions and other preliminaries.

Let G be a topological group which is Hausdorff but not necessarily locally compact. A *representation* U of G is a continuous homomorphism of G into the unitary group of some finite-dimensional Hilbert space H. That is, the word "*representation*" is used to mean "*finite-dimensional continuous unitary representation*". The *dual* $\hat{G}$ of G is the set of all unitary equivalence classes of irreducible representations of G. The group G is said to be *maximally almost periodic* (MAP) if the representations of G separate points in G. The *von Neumann kernel* G_0 of G is the intersection of all kernels of representations of G. Clearly, $G_0=\{e\}$ if and only if G is MAP. The definitions of the terms almost periodic function and compact group Σ associated with G and their relationship with the representations of G are to be found in Dixmier [3, section 16, pp. 296–301]. Another approach to the introduction of the subject is given in [1] and [8]. The following basic facts are immediate consequences of the definition (see [4, lemma 1, p. 150]):

(1) if G is MAP, then every subgroup of G is MAP;

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(2) if N is a normal subgroup of G such that G/N is MAP, then $G_0 \subset N$; and
(3) G_0 equals the intersection of all normal subgroups N of G such that G/N is MAP.

We note that if G/N is MAP, then G/N is Hausdorff so that N must be closed.

We now state one more very useful fact. Let G be a topological group and let H be a closed subgroup of finite index in G. Then:

(i) every continuous almost periodic function f on H can be extended to a continuous almost periodic function $f^{\sim}$ on G by defining $f^{\sim}$ to be zero on each left coset other than H [11, lemma 4];

(ii) there is a closed normal subgroup N of G which has finite index in G and is contained in H (the group constructed in [6, 4.21 (d)] is closed when H is);

(iii) if H (or N) is MAP, then G is MAP (see (2) above); and

(iv) let $h \in H$ and let V be a representation of H such that $V_h \neq I$; then the representation U, induced by V ([5, Theorem 16.7.1] or, for more detail, [13, Theorem 1.5.4]) is a representation of G which separates h from the identity.

Although the class of MAP groups is quite restrictive, (for example, the Freudenthal–Weil theorem which states that a locally compact connected group is MAP if and only if it is the direct product of a vector group (R^n) and a compact group, cf. Weil [12, pp. 127–129] or Dixmier [3, Théorème 16.4.6, p. 303]) it contains all compact groups, all locally compact Abelian groups and all free groups. Furthermore, it follows easily from (2) above that every direct product of MAP groups is again MAP.

We now wish to make a few "well-known" remarks designed to aid the reader in applying the theories of groups with operators and of modules to representation theory.

Let U be a representation of a topological group G with representation space H_U. Then U defines a homomorphism of the group algebra $A(G)$ of G into the endomorphism group of H_U by

$$\sum a_i x_i \to \sum a_i U_{x_i},$$

where $\sum a_i x_i$ is a finite formal sum of complex multiples of elements of the multiplicative group G. Thus H_U is an $A(G)$-module.

Let U and V be irreducible representations of G and assume that there is a module isomorphism T mapping H_U onto H_V. Then there is a $\beta > 0$, such that βT is an isometric isomorphism between the Hilbert spaces H_U and H_V so that U is unitarily equivalent to V. We indicate the proof. It is trivial that T is linear and by definition $TU_x = V_xT$ for each x in G. Taking adjoints we obtain $T^* V_x = U_x T^*$ so that $U_x = T^{-1}T^{*-1}U_xT^*T$. By Schur's lemma, there is a complex number a such that $T^*T = aI$, but since T^*T is positive-definite, a must be greater than zero. Let $\beta = a^{-\frac{1}{2}}$.

Using this fact together with an easy argument and the Krull–Schmidt theorem, the following statement can be justified: If U and V are unitarily equivalent representations of G, say $TU = VT$, then T is an $A(G)$-module isomorphism of H_U onto H_V. Hence, if

$$U = U_1^{(1)} \oplus \ldots \oplus U_n^{(n)} \quad \text{and} \quad V = V^{(1)} \oplus \ldots \oplus V^{(m)},$$

where the $U^{(j)}$ and the $V^{(k)}$ are irreducible, then $n = m$ and there is a permutation τ of $\{1,\ldots,n\}$ such that $U^{(j)}$ is unitarily equivalent to $V^{\tau(j)}$ for each j, $j = 1,\ldots,n$.

We conclude this section with a few more definitions designed to facilitate the investigation of relationships between the representations of a group and those of a normal subgroup. These relationships will then lead us to facts about the structure of MAP groups. Let G be a topological group and let $\sigma \in \hat{G}$. Let T be any set of topological automorphisms of G. For each $t \in T$, we define $t^*\sigma$ to be that element in $\hat{G}$ which, if $U \in \sigma$, has representative $t^*(U) = U \circ t^{-1}$. It is easy to see that this concept is well-defined. The *orbit of σ by T* is the set $\{t^*\sigma : t \in T\}$ written

$$O(\sigma,T) = \{t^*\sigma :\ t \in T\}.$$

Furthermore, we say that *σ is finitely orbited by T* if $O(\sigma,T)$ is finite and denote by $F(\hat{G},T)$ the set of all elements of $\hat{G}$ which are finitely orbited by T. The set $F(\hat{G},T)$ is said to separate points in G if, for every $x \in G$, $x \neq e$, there exists $U \in \sigma \in F(\hat{G},T)$ such that $U_x \neq I$.

We write C for the field of complex numbers and $\mathfrak{U}(n)$ (and $\mathscr{L}(n)$ respectively) for the group of all unitary (and linear respectively) operators on C^n.

2. Fundamentals and semidirect products.

The following result provides the basis for our investigations.

THEOREM 1. *Let G be a normal subgroup of a topological group K. Let T be the subgroup of the group of topological automorphisms of G consisting of the restrictions to G of inner automorphisms of K. That is,*

$T=\{t(x): x \in K\}$ where $t(x)(g)=xgx^{-1}$ for each $g \in G$. If $U \in \sigma \in \hat{K}$ and if $y \in G$ is such that $U_y \neq I$, then there exists an element of $F(\hat{G},T)$ which separates y from the identity. In particular, if K is MAP, *then $F(\hat{G},T)$ separates points in G.*

Proof. Let V be the restriction of U to G. Then V is a representation of G such that $V_y \neq I$. Thus there are irreducible representations $V^{(j)}$, $j=1,\ldots,n$, of G such that

$$V = V^{(1)} \oplus \ldots \oplus V^{(n)} .$$

Furthermore, since $V_y \neq I$, there exists j_0, $1 \leqq j_0 \leqq n$, such that $V_y{}^{(j_0)} \neq I$. Let τ be the equivalence class of $V^{(j_0)}$. The theorem will be proved when we show that $\tau \in F(\hat{G},T)$. For $x \in K$, we have

$$t(x)^* V = t(x)^* V^{(1)} \oplus \ldots \oplus t(x)^* V^{(n)} . \tag{1}$$

To see this let H_j be the representation space of $V^{(j)}$. If $a \in H_j$ and if $z \in G$, then

$$\big(t(x)^* V\big)_z(a) = V_{x^{-1}zx}(a) = V^{(j)}_{x^{-1}zx}(a) = \big(t(x)^* V^{(j)}\big)_z(a) .$$

It is trivial that

$$t(x)^* V = U_x{}^{-1} V U_x , \tag{2}$$

so that $t(x)^* V$ is unitarily equivalent to V. Hence from section 1 we conclude that $t(x)^* V^{(j_0)}$ is unitarily equivalent to one of the $V^{(j)}$, $j=1,\ldots,n$, which implies that $V^{(j_0)} \in \tau \in F(\hat{G},T)$.

Although, in the terminology of theorem 1, the condition that $F(\hat{G},T)$ separate points in G is stronger than assuming that G is MAP, even when the factor group K/G is assumed to be MAP, it is not sufficient to imply that K is MAP. However, if G is a semidirect factor of K, these two conditions imply that K is MAP. These assertions will be proved later in theorem 4 and example 1 of section 3. The following theorem will be a useful tool in the sequel.

Theorem 2. *Let K, G, and T be as in theorem 1. Let $\sigma \in F(\hat{G},T)$. Then the mapping Σ which sends x onto the restriction of $t(x)^*$ to $O(\sigma,T)$ is a continuous homomorphism of K onto a finite group. The kernel of Σ contains G.*

Proof. First, we prove that $t(xy)^* = t(x)^* t(y)^*$. Let U be any representation of G, let $x,y \in K$ and let $z \in G$. Then

$$\big(t(xy)^* U\big)_z = U_{y^{-1}(x^{-1}zx)y} = \big(t(y)^* U\big)_{x^{-1}zx} = \Big(t(x)^*\big(t(y)^* U\big)\Big)_z ,$$

which is equivalent to our assertion. Using this, a trivial computation shows that each $t(x)^*$ is a one to one mapping of $\hat{G}$ into $\hat{G}$, so that, as a consequence, the restriction $\Sigma(x)$ of $t(x)^*$ to $O(\sigma,T)$ must be a permutation of $O(\sigma,T)$. Thus Σ is well-defined and is a homomorphism. Since $\ker\Sigma$ has finite index in K, $\ker\Sigma$ is closed if and only if $\ker\Sigma$ is open. Furthermore, any homomorphism with open kernel is continuous so that it will suffice to prove that $\ker\Sigma$ is closed. Let $\{x_\alpha : \alpha \in A\}$ be a net in $\ker\Sigma$ which converges to some point x in K. Let $U \in \varrho \in O(\sigma,T)$. To each x_α there corresponds a unitary operator M_α such that $t(x_\alpha)^* U = M_\alpha U M_\alpha^{-1}$. Thus $\{M_\alpha : \alpha \in A\}$ is a net in the (compact) unitary group of dimension equal to the dimension of U. Consequently there are a subnet $\{M_\beta : \beta \in B\}$ of $\{M_\alpha : \alpha \in A\}$ and a unitary operator M such that $M_\beta \to M$. Let $\{x_\beta : \beta \in B\}$ be the corresponding subnet of $\{x_\alpha : \alpha \in A\}$. Since $x_\beta \to x$ and since U is continuous, for any $z \in G$ we have

$$\big(t(x_\beta)^* U\big)_z = U_{x_\beta^{-1} z x_\beta} \to U_{x^{-1} z x} = \big(t(x)^* U\big)_z .$$

On the other hand,

$$t(x_\beta)^* U = M_\beta U M_\beta^{-1} \to M\, U M^{-1} .$$

Thus $t(x)^* U$ equals MUM^{-1} and we have proved that

$$\Sigma(x)(\varrho) = t(x)^*(\varrho) = \varrho$$

for each $\varrho \in O(\sigma,T)$ so that $x \in \ker\Sigma$.

For the remainder of this section we will consider the case when K is the semidirect product of groups G and H. That is, $K = GH$ where G is a normal subgroup of K, $G \cap H = \{e\}$ and the topology of K is the product topology $G \times H$. As in theorem 1 above T is the group of restrictions to G of inner automorphisms of K. Below we will prove a series of technical lemmas which will aid us in proving the following fact: if $x \in G$ and if $U \in \varrho \in F(\hat{G},T)$, $U_x \neq I$, then there exists a representation V of K such that $V_x \neq I$. This is the major step required in the proof of the fact that $K = GH$ is MAP if and only if H is MAP and $F(\hat{G},T)$ separates points in G. It is unfortunate that we cannot actually construct the representation V above, but attempts at the construction of such a representation lead directly to consideration of "projective representations" (see Mackey [10]). It is not clear that a more aesthetically pleasing proof can be obtained using such a construction.

Let $U \in \varrho \in \hat{G}$. Then $t(xh)^* U$ is unitarily equivalent to $t(h)^* U$ whenever $x \in G$ and $h \in H$ since

$$t(xh)^* U = U_x^{-1}\big(t(h)^* U\big) U_x .$$

This means that for questions relating to the finiteness of orbits $O(\varrho,T)$ we can restrict our attention to the automorphisms of G defined by elements of H. Furthermore if $U \in \varrho \in F(\hat{G},T)$ and Σ is the mapping of K into the group of permutations of $O(\varrho,T)$ defined in theorem 2, then it follows from $G \subset \ker\Sigma$ that $\ker\Sigma$ is the semidirect product of G and a subgroup, M, of H. Here M is the projection of $\ker\Sigma$ into H and M is a closed (open) subgroup of finite index in H. Having defined this subgroup M, which depends on $\varrho \in F(\hat{G},T)$, for a fixed $U \in \varrho$ and for each $h \in M \subset \ker\Sigma$ we define $W_h \in \mathfrak{U}(n)$ by the equation

$$t(h^{-1})^* U = W_h U W_h^{-1} .$$

Also for any $W \in \mathfrak{U}(n)$, we define $\mathscr{I}(W)$ to be the operator on $\mathscr{L}(n)$ with the property: if $A \in \mathscr{L}(n)$, then $\mathscr{I}(W)(A) = WAW^{-1}$. It is clear that $\mathscr{I}(W)$ is linear, that is, $\mathscr{I}(W) \in \mathscr{L}(n^2)$. We note that $\mathscr{L}(n)$ is a Hilbert space with an inner product defined by

$$(1) \qquad \langle A,B\rangle = \operatorname{tr} B^*A \quad \text{for} \quad A,B \in \mathscr{L}(n) ,$$

where, as usual, B^* is the adjoint of the operator B and tr is the trace function on $\mathscr{L}(n)$.

Lemma 1. *Let notation be as in the preceding paragraph. The mapping $h \to \mathscr{I}(W_h)$ is a well-defined continuous homomorphism of M into $\mathfrak{U}(n^2)$, the unitary group in $\mathscr{L}(n^2)$.*

Proof. First we prove that $\mathscr{I}(W_h) \in \mathfrak{U}(n^2)$ for each $h \in M$. By (1), for each $A \in \mathscr{L}(n)$, we have

$$\begin{aligned} \langle \mathscr{I}(W_h)(A), \mathscr{I}(W_h)(A)\rangle &= \operatorname{tr}((W_h A W_h^{-1})^* W_h A W_h^{-1}) \\ &= \operatorname{tr}(W_h A^* W_h^{-1} W_h A W_h^{-1}) \\ &= \operatorname{tr}(W_h A^* A W_h^{-1}) \\ &= \operatorname{tr}(A^*A) = \langle A,A\rangle . \end{aligned}$$

Hence $\mathscr{I}(W_h) \in \mathfrak{U}(n^2)$.

Next we show that $h \to \mathscr{I}(W_h)$ is a homomorphism. Let $h,k \in M$ and let x be an arbitrary element of G, then $\mathscr{I}(W_{hk})(U_x) = W_{hk} U_x W_{hk}^{-1}$. But

$$\begin{aligned} W_{hk} U_x W_{hk}^{-1} = (t((hk)^{-1})^* U)_x &= (t(k^{-1})^* t(h^{-1})^* U)_x \\ &= (t(k^{-1})^* W_h U W_h^{-1})_x \\ &= (W_h U W_h^{-1})_{kxk-1} \\ &= W_h U_{kxk-1} W_h^{-1} \\ &= (W_h t(k^{-1})^* U W_h^{-1})_x \\ &= \mathscr{I}(W_h)[\mathscr{I}(W_k) U_x] . \end{aligned}$$

By assumption U is an irreducible representation of G so that $\{U_x\colon x\in G\}$ spans $\mathscr{L}(n)$. This is a special case of Burnside's theorem; see Jacobson [9, page 276]. We have just proved that $\mathscr{I}(W_{hk})(U_x)=\mathscr{I}(W_h)\mathscr{I}(W_k)(U_x)$ for each $x\in G$ so by linearity we have $\mathscr{I}(W_{hk})(A)=\mathscr{I}(W_h)\mathscr{I}(W_k)(A)$ for each $A\in\mathscr{L}(n)$ and $\mathscr{I}(W_{hk})=\mathscr{I}(W_h)\mathscr{I}(W_k)$. This proves that $h\to\mathscr{I}(W_h)$ is a homomorphism of M into $\mathfrak{U}(n^2)$. The continuity of this homomorphism is proved as follows. Since multiplication is continuous and U is a representation, the mapping of $G\times M$ into $\mathfrak{U}(n)$ defined by

$$(x,h)\to U_{hxh^{-1}}=\left(t(h^{-1})^*U\right)_x$$

is continuous. Let $\{h_\alpha\colon \alpha\in A\}$ be a net in M which converges to the identity e in M. Then, for each fixed $x\in G$,

$$\mathscr{I}(W_{h_\alpha})(U_x)=\left(t(h_\alpha{}^{-1})^*U\right)_x\to\left(t(e)^*U\right)_x=U_x\,.$$

Applying Burnside's theorem again, we obtain $\mathscr{I}(W_{h_\alpha})(A)\to A$ for each $A\in\mathscr{L}(n)$. That is, $\mathscr{I}(W_{h_\alpha})\to I$ where I is the identity operator on $\mathscr{L}(n^2)$. Consequently the homomorphism $h\to\mathscr{I}(W_h)$ is continuous.

We now define $\mathfrak{A}$ to be the closure in $\mathfrak{U}(n^2)$ of $\{\mathscr{I}(W_h)\colon h\in M\}$. Since $\mathfrak{A}$ is the closure of a homomorphic image of M in a compact group, $\mathfrak{A}$ is a compact group.

LEMMA 2. *Every element of $\mathfrak{A}$ is a "unitary conjugation operator", that is, if $\mathscr{J}\in\mathfrak{A}$, then there exists $W\in\mathfrak{U}(n)$ such that $\mathscr{J}(B)=WBW^{-1}=\mathscr{I}(W)(B)$ for each $B\in\mathscr{L}(n)$.*

PROOF. If $\mathscr{J}\in\mathfrak{A}$, then there exists a sequence of elements of the form $\mathscr{I}(W_h)$ converging to $\mathscr{J}$. Now use the compactness of $\mathfrak{U}(n)$ to define W as the limit of a subsequence of the W_h's. An application of the Burnside theorem yields the result.

From the above lemma it is clear that $\mathfrak{A}$ is a subgroup of the group of topological automorphisms of $\mathfrak{U}(n)$. Thus we can define a semidirect product $\mathfrak{U}(n)\circledS\mathfrak{A}$ with multiplication

$$\begin{aligned}(U_1,\mathscr{I}(W_1))(U_2,\mathscr{I}(W_2))&=(U_1,\mathscr{I}(W_1)(U_2),\mathscr{I}(W_1)\circ\mathscr{I}(W_2))\\&=(U_1W_1U_2W_1{}^{-1},\mathscr{I}(W_1W_2))\,.\end{aligned}$$

LEMMA 3. *Using the same notation as above, $\mathfrak{U}(n)\circledS\mathfrak{A}$ is a topological group.*

PROOF. It suffices to show that the mapping of $\mathfrak{U}(n)\times\mathfrak{A}\to\mathfrak{U}(n)$ defined by $(V,\mathscr{I}(W))\to\mathscr{I}(W)(V)$ is continuous. (See [7, p. 42].) Let

$(V_l)_{l=1}^{\infty}$ and $(\mathscr{I}(W_m))_{m=1}^{\infty}$ be sequences in $\mathfrak{U}(n)$ and $\mathfrak{A}$ respectively which converge to V and $\mathscr{I}(W)$ respectively. Since $\mathfrak{U}(n)$ is a topological group, we have

$$\lim_{l,m}\mathscr{I}(W_m)(V_l) = \lim_{l,m}W_mV_lW_m{}^{-1} = WVW^{-1} = \mathscr{I}(W)(V) .$$

We are now ready to state and prove the theorem towards which we have been aiming throughout this section.

THEOREM 3. *Let $K=GH$ be a semidirect product of a normal subgroup G and a subgroup H. Let $U \in \varrho \in F(\hat{G},T)$ where T is as in theorem* 1. *Let n be the dimension of U. Then there are*

(i) *an open subgroup GM of GH such that $[GH: GM] < \infty$,*

(ii) *a compact topological group $\mathfrak{U}(n) \circledS \mathfrak{A}$ and*

(iii) *a continuous homomorphism φ mapping GM into $\mathfrak{U}(n) \circledS \mathfrak{A}$ defined by $\varphi(xh)=(U_x, \mathscr{I}(W_h))$ where $x \in G$ and $h \in M$.*

PROOF. Statements (i) and (ii) were proved above so we proceed to statement (iii). The mapping φ is a homomorphism, since, if $x,y \in G$ and $h,k \in M$, by the definitions and lemma 1 we have

$$\begin{aligned}\varphi(xhyk) = \varphi((xhyh^{-1})(hk)) &= (U_xU_{hyh^{-1}}, \mathscr{I}(W_{hk}))\\ &= \big(U_x(t(h^{-1})^*U)_y, \mathscr{I}(W_h)\mathscr{I}(W_k)\big)\\ &= (U_x\mathscr{I}(W_h)(U_y), \mathscr{I}(W_h)\mathscr{I}(W_k))\\ &= (U_x, \mathscr{I}(W_h))(U_y, \mathscr{I}(W_k)) .\end{aligned}$$

To prove that φ is continuous, it suffices to show that φ is continuous at (e,e). Since φ is a mapping into a topological product, it suffices to show that the composition of φ with each of the projection mappings is continuous. That is, the mapping of GM into $\mathfrak{U}(n)$ defined by $(x,h) \to U_x$ should be continuous — which it is, since U is a continuous representation and the mapping of $G \circledS M$ into $\mathfrak{A}$ defined by $(x,h) \to \mathscr{I}(W_h)$ should be continuous — which it is by lemma 1.

THEOREM 4. *Let $K=GH$ and T be as in theorem* 3. *Let H_0 (respectively K_0) be the von Neumann kernel of H (respectively K). Let*

$$S = \bigcap \{\ker U : U \in \sigma \in F(\hat{G},T)\} .$$

Then $K_0=SH_0$ is a semidirect product. In particular, K is MAP *if and only if H is* MAP *and the finitely orbited representations of G separate points in G.*

PROOF. Assume $xh \notin K_0$ but $x \in G$, $h \in H$, then there exists a representation W of K such that $W_{xh} \neq I$. There are two cases. Case (1). If

$h \in H_0$, then $h^{-1} \in H_0$, and the restriction of W to H_0 must be trivial so that

$$I \neq W_{xh} I = W_{xh} W_{h^{-1}} = W_x .$$

From theorem 1 it follows that $x \notin S$ so that $xh \notin SH_0$. Case (2). If $h \notin H_0$, then $xh \notin SH_0$.

Now assume $xh \notin SH_0$. There are again two cases. First, if $h \in H_0$, then $x \notin S$, so there exists $U \in \varrho \in F(\hat{G},T)$ such that $U_x \neq I$. We now apply theorem 3 to this representation U. Since $\mathfrak{U}(n) \circledS \mathfrak{A}$ is compact, there is a representation of $\mathfrak{U}(n) \circledS \mathfrak{A}$ which separates $(U_x, \mathscr{I}(W_e)) = \varphi(x,e)$ from the identity in $\mathfrak{U}(n) \circledS \mathfrak{A}$. The composition of φ with this representation is a representation of GM which separates x from the identity in GM. As noted in section 1 this representation can be extended to a representation Y of $K = GH$. As in case (1) above $Y_h = I$ so that $I \neq Y_x = Y_{xh}$. That is, Y separates xh from the identity so that $xh \notin K_0$. Finally, if $h \notin H_0$, then there exists a representation V of H such that $V_h \neq I$. The composition of the projection of K into H with V is a representation of K which separates xh from the identity.

3. Groups with Abelian normal subgroups.

Since equivalent one-dimensional representations are equal, we identify the dual $\hat{G}$ of an Abelian group G with the character group X of G. Thus we note that theorem 4 has an immediate analogue for Abelian groups with "$F(X,T)$" replacing "$F(\hat{G},T)$". If G is not locally compact, we give X the discrete topology and otherwise X has the usual topology which has all sets

$$P(F,\varepsilon) = \{\chi \in X : |\chi(x) - 1| < \varepsilon \text{ for all } x \in F\}$$

with F compact in G and $\varepsilon > 0$ as a basis at the identity character 1. See Hewitt and Ross [6]. We note that when G is abelian, theorem 4 is much easier to prove as a finitely orbited character can be extended to the group GM of theorem 3 by making it constant on cosets xM, $x \in G$, and then the representation induced from this extension has the properties we need. We state an easy lemma whose verification is left to the reader.

LEMMA 4. *Let G be an Abelian normal subgroup of a topological group K. As in theorem 1, let T be the set $\{t(x) : x \in K\}$ where $t(x)g = xgx^{-1}$ for $x \in K$ and $g \in G$. Let X be the character group of G. Then $F(X,T)$ is a subgroup of X.*

We note that when G is locally compact, the second annihilator of a subgroup of X is its closure. (See Bourbaki [2, corollaire 1, p. 125] or Hewitt and Ross [6, 23.24 (a) and 24.10].) From this it is immediate that a subgroup of X separates points in G if and only if it is dense in X. From these remarks it follows that we have the following corollary to theorem 4.

THEOREM 5. *Let $K=GH$ be a semidirect product of a locally compact Abelian normal sub-group G and a subgroup H. Then the following statements are equivalent:*
(i) *$K=GH$ is* MAP;
(ii) *$F(X,T)$ separates points in G and H is* MAP;
(iii) *$F(X,T)$ is dense in X and H is* MAP.

The following result is useful when considering examples.

THEOREM 6. *Let G be an Abelian normal subgroup of a topological group K and let C be the centralizer of G in K. If K is* MAP, *then C contains the intersection of closed (open) normal subgroups of finite index in K.*

PROOF. Let $y \notin C$, then there is $x \in G$ such that $e \neq z = x^{-1}yxy^{-1}$. Since G is normal, $z \in G$. By theorem 1 there exists $\chi \in F(X,T)$ such that $\chi(z) \neq 1$. Now

$$1 \neq \chi(z) = \chi(x^{-1})\chi(yxy^{-1}),$$

so

$$\chi(x) \neq \chi(yxy^{-1}) = t(y^{-1})^*(\chi)(x) .$$

That is, $\chi \neq t(y^{-1})^*(\chi)$. We now apply theorem 2. Thus $y^{-1} \notin \ker\Sigma$ and hence $y \notin \ker\Sigma$ where Σ is the mapping which sends $w \in K$ to the restriction of $t^*(w)$ to $O(\chi,T)$. Furthermore $\ker\Sigma$ is a closed normal subgroup of K with finite index in K.

The following corollary is of particular interest when H is connected.

COROLLARY 1. *Let $K=GH$ be a semidirect product of an Abelian normal subgroup and a subgroup H. Assume that H contains no proper open subgroups of finite index. Then the set*

$$A = \{z \in G : z = xhx^{-1}h^{-1} \text{ for some } x \in G \text{ and } h \in H\}$$

is contained in the von Neumann kernel K_0 of K. Moreover, if K is MAP, *then K is actually the direct product of G and H.*

PROOF. If $z \in A$, then $z \neq e$ implies that h is not in the centralizer

C of G in K where $z=xhx^{-1}h^{-1}$. Thus, as in the proof of theorem 6, if there is a representation of K which separates z from the identity, there is an open normal subgroup of K which does not contain h. The projection of this subgroup into H is an open normal subgroup of H with finite index in H which is a contradiction. Hence $z \in K_0$. If K is MAP, then $K_0=\{e\}$ so that $A=\{e\}$ and the elements of H commute with those of G. Thus K is a direct product.

The author wishes to express his thanks to Professor Lewis Robertson for bringing the following example to his attention.

EXAMPLE 1. In view of theorem 4 it is important to note that there exist groups K with non-trivial normal Abelian subgroups G such that G is MAP, the finitely orbited representations of G separate points in G, but K is not MAP. Let K be the group of all matrices of the form

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \quad \text{where } a,\ b, \text{ and } c \text{ are in } \mathsf{R}.$$

Let G be the set of all

$$\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{where } c \in \mathsf{R}.$$

Clearly G is isomorphic to R and by direct computation one verifies that

(i) G is the center of K and

(ii) $K/G=\mathsf{R}^2$.

However, K is not MAP, as the following remarks show. Let $\mathfrak{S}(\mathsf{R}^2)$ be the group of topological automorphisms of R^2. We show that K is a semidirect product of R^2 and R. Our notation is taken from Hewitt and Ross [6, p. 6]. Define $\beta\colon \mathsf{R} \to \mathfrak{S}(\mathsf{R}^2)$ by $\beta(a)(z,y)=(z+ay,y)$. Now let $(c,b), (z,y) \in \mathsf{R}^2$ and let $a,x \in \mathsf{R}$. Then, using the multiplication of $\mathsf{R}^2 \circledS_\beta \mathsf{R}$, we have

$$\begin{aligned} ((c,b),\, a)((z,y),\, x) &= ((c,b)+\beta(a)(z,y),\, a+x) \\ &= ((c+z+ay,\, b+y),\, a+x)\,. \end{aligned}$$

From this it is easy to verify that the mapping

$$((c,b),\, a) \;\to\; \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

is a topological isomorphism of $\mathsf{R}^2 \circledS_\beta \mathsf{R}$ onto K. Since R is connected, $\mathsf{R}^2 \circledS_\beta \mathsf{R}$ satisfies the hypotheses of corollary 1. Hence, if $\mathsf{R}^2 \circledS_\beta \mathsf{R}$ were MAP, then $\mathsf{R}^2 \circledS_\beta \mathsf{R}$ would be a direct product, which is clearly not the case. Thus $\mathsf{R}^2 \circledS_\beta \mathsf{R}$ and K are not MAP.

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