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HYPERSPACES OF PROXIMITY SPACES

LOUIS J. NACHMAN

In [4] Hausdorff defines a metric for the collection of all closed bounded subsets of a metric space. Various methods for defining topologies on the power set of a given topological space can be found in [8]. More recently several authors have been concerned with uniformities generated on the power set of a given uniform space [3], [5], [11]. It is the purpose of this paper to define and study two proximities on the power set of a given proximity space. The second part of this paper is devoted to this study. The first part, roughly sections 1–4, is a synopsis of material needed for the discussion in the second part.

We assume that the reader is familiar with the definition and basic properties of a proximity space (p-space) (X,δ), its proximal neighborhood relation $\Subset$, its topology $\mathfrak{J}(\delta)$, and proximally continuous functions (see for example [10]). In particular we do not assume that a p-space (X,δ) must be separated.

1. Uniform spaces and proximity spaces.

If $(X,\mathscr{U})$ is a uniform space, then it is possible to define a p-relation $\delta(\mathscr{U})$ on X as follows:

$$A\,\delta(\mathscr{U})B \quad \text{iff} \quad U[A]\cap B \neq \emptyset \quad \text{for all } U\in\mathscr{U}\,.$$

For any p-space (X,δ) we denote by $\pi(\delta)$ the class of all uniformities $\mathscr{U}$ on X for which $\delta(\mathscr{U})=\delta$.

For any p-space (X,δ) we define $\mathscr{S}(\delta)$ to be $\bigcup\{\mathscr{U} : \mathscr{U}\in\pi(\delta)\}$. The following facts are almost obvious and their verification is left to the reader.

THEOREM 1. *Suppose (X,δ) and (Y,η) are p-spaces.*

(1) *If $f\colon X\to Y$, then f is p-continuous iff $V\in\mathscr{S}(\eta)$ implies*
$$(f\times f)^{-1}[V]\in\mathscr{S}(\delta)\,.$$

(2) *If $A,B\subseteq X$, then $A\Subset B$ iff there is a $U\in\mathscr{S}(\delta)$ such that $U[A]\subseteq B$.*

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(3) *If* $A \subseteqq X$ *and* $\delta|A$ *is the subspace proximity, then*

$$\mathscr{S}(\delta) \cap (A \times A) \subseteqq \mathscr{S}(\delta|A)$$

and they are generally not equal.

(4) *If* $O \subseteqq X$, *then* $O \in \mathfrak{J}(\delta)$ *iff for each* $x \in O$ *there is a* $U \in \mathscr{S}(\delta)$ *such that* $U[x] \subseteqq O$.

If (X,δ) is a p-space let $\mathscr{U}(\delta) = \inf \pi(\delta)$. It is well known that $\mathscr{U}(\delta)$ is a totally bounded uniformity and that $\mathscr{U}(\delta) \in \pi(\delta)$.

There is a categorical isomorphism φ from the category of p-spaces and p-continuous functions to the category of totally bounded uniform spaces and uniformly continuous functions. For any object (X,δ), $\varphi[(X,\delta)]$ is $(X,\mathscr{U}(\delta))$. If

$$f:\ (X,\delta) \to (Y,\eta)$$

is p-continuous, then

$$\varphi(f):\ (X,\mathscr{U}(\delta)) \to (Y,\mathscr{U}(\eta))$$

is uniformly continuous where $\varphi(f)(x) = f(x)$.

2. The lattice of proximities.

Suppose X is a set. Let $\boldsymbol{P}(X)$ denote the class of all proximities for the set X. A reflexive, antisymmetric, transitive ordering is defined on $\boldsymbol{P}(X)$ which makes the functor φ mentioned above an order isomorphism. If δ_1 and δ_2 are in $\boldsymbol{P}(X)$ then

$$\delta_1 \leqq \delta_2 \quad \text{iff} \quad A\delta_2 B \text{ implies } A\delta_1 B\,.$$

It is a routine matter to prove the following theorem.

THEOREM 2. *Suppose* δ_1 *and* δ_2 *are in* $\boldsymbol{P}(X)$. *Then the following are equivalent*:

(1) $\delta_1 \leqq \delta_2$,
(2) $A\,\delta_2 B$ *implies* $A\,\delta_1 B$,
(3) $A\,\bar{\delta}_1 B$ *implies* $A\,\bar{\delta}_2 B$ ($\bar{\delta}$ *denotes "not* δ"),
(4) $A \Subset_1 B$ *implies* $A \Subset_2 B$,
(5) $\mathscr{U}(\delta_1) \subseteqq \mathscr{U}(\delta_2)$,
(6) $\mathscr{S}(\delta_1) \subseteqq \mathscr{S}(\delta_2)$.

Condition (5) above and the fact that, if $\mathscr{U}_1$ and $\mathscr{U}_2$ are uniformities for X such that $\mathscr{U}_1 \subseteqq \mathscr{U}_2$, then $\delta(\mathscr{U}_1) \leqq \delta(\mathscr{U}_2)$ establish the order isomorphism mentioned earlier.

If $\mathscr{F}$ is a non-empty collection of p-relations on a set X, it is possible to characterize the supremum of $\mathscr{F}$ in terms of the elements of $\mathscr{F}$. This characterization is used in the literature although not (to the author's knowledge) identified. If

$$A = \bigcup\{A_i : 1 \leqq i \leqq n\}$$

we will call $\{A_i\}$ a finite cover from below of A.

THEOREM 3. *Suppose X is a set and $\{\delta_\alpha : \alpha \in I\}$ is a non-empty family of proximities for X. Define δ^* as follows: $A\,\delta^*B$ iff for any $\{A_i\}$ and $\{B_j\}$ which are finite covers from below of A and B respectively there is an i^* and a j^* such that $A_{i^*}\delta_\alpha B_{j^*}$ for all $\alpha \in I$. Then δ^* is a p-relation, $\delta^* \geqq \delta_\alpha$ for all $\alpha \in I$, and $\delta \geqq \delta_\alpha$ for all $\alpha \in I$ implies $\delta \geqq \delta^*$.*

The proof of this result is left to the reader.

For any set X, the class $\boldsymbol{P}(X)$ has a largest element δ_M and a smallest element δ_m:

$$\begin{aligned} A\,\delta_M B \quad &\text{iff} \quad A \cap B \neq \emptyset\,, \\ A\,\delta_m B \quad &\text{iff} \quad A \neq \emptyset \neq B\,. \end{aligned}$$

Thus $(\boldsymbol{P}(X), \leqq)$ is a complete lattice.

3. Complete proximity spaces.

Suppose (X,δ) is a p-space and $S: D \to X$ is a net. S will be called *proximally Cauchy* (p-Cauchy) iff for every $U \in \mathscr{S}(\delta)$ there is a $d = d(U) \in D$ such that $d_1, d_2 \geqq d$ implies $(S(d_1), S(d_2)) \in U$. A proximity space is complete if every p-Cauchy net converges.

The literature on complete proximity spaces is substantial and we will not restate all the known results here. Later we will use the fact that, if $(X,\mathscr{U})$ is a complete uniform space, then $(X,\delta(\mathscr{U}))$ is a complete proximity space. A corollary to the following lemma is useful in constructing counter-examples. If $S: D \to X$ is a net, we will denote a subnet of S by $S \circ j: E \to X$ where it is understood that $j: E \to D$ has the subnet property (see Kelley [6]).

LEMMA 1. *Suppose (X,δ) is a p-space and $S: D \to X$ is a net, D a linearly ordered set. Then S is p-Cauchy if*

$$S[j_1[E_1]]\ \delta\ S[j_2[E_2]]$$

for every pair of subnets $S \circ j_i: E_i \to X$, $i = 1, 2$.

PROOF. Suppose S is not p-Cauchy. Then there is a $U \in \mathscr{S}(\delta)$, U symmetric, such that for any $d \in D$ there are indices $d_i(d) \geqq d$, $i = 1, 2$, such that $\big(S(d_1(d)), S(d_2(d))\big)$ is not in U. Define a net $T: D \to X \times X$ so that

$$T(d) = \big(S(d_1(d)), S(d_2(d))\big) .$$

Let U^* be a symmetric element of $\mathscr{S}(\delta)$ such that $U^* \circ U^* \circ U^* \circ U^* \subseteqq U$. By a result of Alfsen and Njåstad [1], there is a subnet $T \circ j: E \to X \times X$ such that

$$\Big(\pi_1\big(T(j(e_1))\big), \pi_2\big(T(j(e_2))\big)\Big) \notin U^* \quad \text{for any} \quad e_1, e_2 \in E$$

(π_i is the ith projection from $X \times X$ to X).

Let $k_i: E \to D$ such that $k_i(e) = d_i(j(e))$, $i = 1, 2$; then the k_i's have the subnet property. Further

$$\pi_i \circ T \circ j = S \circ k_i ;$$

thus the

$$\pi_i \circ T \circ j : E \to X$$

are subnets of S. Since $\big(\pi_1(T(j(e_1))), \pi_2(T(j(e_2)))\big) \in U^*$ for no $e_1, e_2 \in E$,

$$\pi_1\big[T[j[E]]\big] \,\bar{\delta}\, \pi_2\big[T[j[E]]\big] .$$

Hence, S not p-Cauchy implies the existence of subnets whose ranges are far as subsets of X.

We note in passing that the converse of lemma 1 is true for any p-Cauchy net and that lemma 1 will not serve to characterize all p-Cauchy nets.

COROLLARY. *Suppose $S: D \to X$ is a net, D a linearly ordered set. Suppose (X, δ) is a p-space and $\mathscr{U} \in \pi(\delta)$. If S is Cauchy in $(X, \mathscr{U})$, then S is p-Cauchy.*

PROOF. If S is Cauchy in $(X, \mathscr{U})$, then the ranges of any pair of subnets of S must be $\delta(\mathscr{U})$ near in X.

4. Hyperuniform spaces.

Let $\mathfrak{P}(X) = \{A : A \subseteqq X\}$ for any set X. Suppose X is a set and $S \subseteqq X \times X$. We define

$$H(S) = \{(A, B) \in \mathfrak{P}(X) \times \mathfrak{P}(X) : A \subseteqq S[B] \text{ and } B \subseteqq S[A]\} .$$

If $(X, \mathscr{U})$ is a uniform space and $\mathfrak{B}$ is a base for $\mathscr{U}$, $\{H(B) : B \in \mathfrak{B}\}$ is a base for a uniformity for $\mathfrak{P}(X)$. Since all bases for $\mathscr{U}$ generate the same uniformity on $\mathfrak{P}(X)$, we denote this uniformity by $H(\mathscr{U})$. The space $(\mathfrak{P}(X), H(\mathscr{U}))$ is called the *hyperuniform space* for $(X, \mathscr{U})$.

If $(X,\mathscr{U})$ is totally bounded, then $(\mathcal{P}(X), H(\mathscr{U}))$ is also totally bounded. The interested reader is referred to [2] or [5] for further information on hyperuniform spaces.

5. Hyperproximity spaces.

Suppose (X,δ) is a proximity space. Because of the functor φ a natural way to induce a proximity relation on $\mathcal{P}(X)$ is to choose the proximity of the hyperuniformity $H(\mathscr{U}(\delta))$. We will denote this hyperproximity by $H_w(\delta)$ and call it the *weak hyperproximity* for (X,δ). The following theorem summarizes some of the properties of $H_w(\delta)$.

THEOREM 4. *Suppose (X,δ) is a p-space.*

(1) *The map $i: X \to \mathcal{P}(X)$ where $i(x)=\{x\}$ is a proximal isomorphism into $(\mathcal{P}(X), H_w(\delta))$.*

(2) *Define a correspondence H_w on the category of p-spaces so that $H_w[(X,\delta)]=(\mathcal{P}(X), H_w(\delta))$ and so that if $f: X \to Y$, then*

$$H_w(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$$

where $H_w(f)(A)=f[A]$ if $A \subseteq X$. Then H_w is a covariant functor.

(3) $\mathscr{U}(H_w(\delta))=H(\mathscr{U}(\delta))$.

The proofs of these results are routine and are left to the reader. In [2] Caulfield states the following result: If $(X,\mathscr{U})$ is uniform and

$$g((x,y)) = \{\{x\}, \{x,y\}\},$$

then g is a uniform isomorphism from $(X\times X, \mathscr{U}\times\mathscr{U})$ into $(\mathcal{P}(\mathcal{P}(X)), H(H(\mathscr{U})))$.

Although the definition for $H_w(\delta)$ is natural, it has certain deficiencies. If (X,δ) is a pseudometrizable p-space, then $(\mathcal{P}(X), H_w(\delta))$ need not be pseudometrizable. We show this using the lemma of Caulfield mentioned above. Suppose the conjecture

"(X,δ) pseudometrizable implies $(\mathcal{P}(X), H_w(\delta))$ pseudometrizable"

is correct. Then it will also be true that (X,δ) pseudometrizable implies $(\mathcal{P}(\mathcal{P}(X)), H_w(H_w(\delta)))$ is pseudometrizable. Since $\mathscr{U}(H_w(\delta))=H(\mathscr{U}(\delta))$,

$$H(H(\mathscr{U}(\delta))) = \mathscr{U}(H_w(H_w(\delta))) .$$

Using Caulfield's lemma, we conclude that, if $(X\times X, \mathscr{U}(\delta)\times\mathscr{U}(\delta))$ is uniformly isomorphic to a subspace of $(\mathcal{P}(\mathcal{P}(X)), H(H(\mathscr{U}(\delta))))$, then $(X\times X, \delta(\mathscr{U}(\delta)\times\mathscr{U}(\delta)))$ is proximally isomorphic to a subspace of

$\big(\mathcal{P}(\mathcal{P}(X)), H_w(H_w(\delta))\big)$. Suppose (X,δ) is pseudometrizable. Then $\big(\mathcal{P}(\mathcal{P}(X)), H_w(H_w(\delta))\big)$ and hence $\big(X \times X, \delta(\mathscr{U}(\delta) \times \mathscr{U}(\delta))\big)$ will be pseudometrizable. Since this is known to be false (see [7]) the original conjecture must be false.

In an attempt to remedy this defect we introduce the following definition. Suppose (X,δ) is a p-space. Define

$$H_s(\delta) = \sup\{\delta(H(\mathscr{U})) : \mathscr{U} \in \pi(\delta)\} .$$

$(\mathcal{P}(X), H_s(\delta))$ will be called the *strong hyperproximity* space for (X,δ).

It is not difficult to show that results similar to (1) and (2) of theorem 4 hold for $H_s(\delta)$.

A reasonable conjecture at this point might be that $\mathscr{S}(H_s(\delta)) = H(\mathscr{S}(\delta))$ where $H(\mathscr{S}(\delta))$ is the collection of all subsets $\mathscr{Q}$ of $\mathcal{P}(X) \times \mathcal{P}(X)$ such that $H(U) \subseteqq \mathscr{Q}$ for some $U \in \mathscr{S}(\delta)$. The author doubts that this is true although we do have

LEMMA 2. *The inclusion $H(\mathscr{S}(\delta)) \subseteqq \mathscr{S}(H_s(\delta))$ holds.*

PROOF. Let $U \in \mathscr{S}(\delta)$. Then there is a $\mathscr{U} \in \pi(\delta)$ such that $U \in \mathscr{U}$. Thus $H(U) \in H(\mathscr{U})$. Since $\delta(H(\mathscr{U})) \leqq H_s(\delta)$,

$$H(\mathscr{U}) \vee \mathscr{U}(H_s(\delta)) \in \pi(H_s(\delta))$$

(see [10]) and hence

$$H(U) \in H(\mathscr{U}) \vee \mathscr{U}(H_s(\delta)) \subseteqq \mathscr{S}(H_s(\delta)) .$$

Generally the two hyperproximities $H_s(\delta)$ and $H_w(\delta)$ will not be equal. The following theorem gives the relationship between the two proximities. A p-space (X,δ) is *completely bounded* ([9]) iff $U \in \mathscr{S}(\delta)$ implies the existence of an $A \subseteqq X$, A finite, such that $U[A] = X$.

THEOREM 5. *For any p-space (X,δ), $H_w(\delta) \leqq H_s(\delta)$. Further, $H_s(\delta) = H_w(\delta)$ iff (X,δ) is completely bounded.*

PROOF. It is obvious that $H_w(\delta) \leqq H_s(\delta)$. That equality holds if (X,δ) is completely bounded is also obvious if we remark that (X,δ) is completely bounded iff $\mathscr{U}(\delta) = \mathscr{S}(\delta)$.

If (X,δ) is not completely bounded, then there is a $U^* \in \mathscr{S}(\delta)$ such that $U^*[A] = X$ for no finite $A \subseteqq X$. Let

$$\mathscr{F} = \{A : A \subseteqq X,\ A \text{ is finite}\} .$$

It follows directly from lemma 2 and (2) of theorem 1 that $H(U^*)[\mathscr{F}]$ is an $H_s(\delta)$ proximal neighborhood of $\mathscr{F}$.

We now claim that $H(U^*)[\mathscr{F}]$ is not an $H_w(\delta)$ proximal neighborhood of $\mathscr{F}$, showing that $H_w(\delta) \neq H_s(\delta)$. Suppose $H(U^*)[\mathscr{F}]$ is an $H_w(\delta)$ proximal neighborhood of $\mathscr{F}$. Then there is a $U \in \mathscr{U}(\delta)$ such that $H(U)[\mathscr{F}]$ is contained in $H(U^*)[\mathscr{F}]$. Since $\mathscr{U}(\delta)$ is totally bounded there is an $A \in \mathscr{F}$ such that $U[A]=X$. Since $A \subseteqq U[X]$ and $X \subseteqq U[A]$,

$$X \in H(U)[\mathscr{F}]$$

and hence

$$X \in H(U^*)[\mathscr{F}] .$$

But then there must be an $A^* \in \mathscr{F}$ such that $X \subseteqq U^*[A^*]$. Since this is not possible, $H(U^*)[\mathscr{F}]$ is not an $H_w(\delta)$ proximal neighborhood of $\mathscr{F}$.

We close this section with the proof that $(\mathcal{D}(X), H_s(\delta))$ is pseudometrizable when (X,δ) is pseudometrizable. Abusing slightly the notation of Alfsen and Njåstad [1] we will call a p-space (X,δ) total iff $\mathscr{S}(\delta)$ is a uniformity.

THEOREM 6. *Suppose (X,δ) is total. Then $H(\mathscr{S}(\delta))$ is a uniformity and $\delta(H(\mathscr{S}(\delta)))=H_s(\delta)$.*

PROOF. From $\mathscr{S}(\delta) \in \pi(\delta)$ it follows that $H_s(\delta) \geqq \delta(H(\mathscr{S}(\delta)))$. Suppose $\mathcal{A}, \mathcal{B} \subseteqq \mathcal{D}(X)$ and $\mathcal{A}\,\delta(H(\mathscr{S}(\delta)))\,\mathcal{B}$. Let $\{\mathcal{A}_i\}$ and $\{\mathcal{B}_j\}$ be finite covers from below of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then $\mathcal{A}_{i^*}\,\delta(H(\mathscr{S}(\delta)))\,\mathcal{B}_{j^*}$ for some i^* and j^*. It follows that

$$H(U)[\mathcal{A}_{i^*}] \cap \mathcal{B}_{j^*} \neq \emptyset$$

for any U in any $\mathscr{U}$ in $\pi(\delta)$. Thus $\mathcal{A}_{i^*}\,\delta(H(\mathscr{U}))\,\mathcal{B}_{j^*}$ for all $\mathscr{U} \in \pi(\delta)$ and $\mathcal{A}\,H_s(\delta)\,\mathcal{B}$.

It is well known that, if (X,δ) is pseudometrizable, (X,δ) is total. Further, if $\mathscr{U}_d$ is the uniformity for the pseudometric, $\mathscr{S}(\delta)=\mathscr{U}_d$. Hence $H_s(\delta)=\delta(H(\mathscr{U}_d))$ in this case. Since $H(\mathscr{U}_d)$ has a countable base it is pseudometrizable and hence so is $H_s(\delta)$. Thus we have proved:

COROLLARY. *Suppose (X,δ) is pseudometrizable. Then $(\mathcal{D}(X), H_s(\delta))$ is pseudometrizable.*

6. Properties of Hyperproximities.

In this section we indicate some of the properties of the two hyperproximities and point out some of the unanswered questions. We remark that if (X,δ) is discrete, that is, $\delta=\delta_M$, then $\Delta_X \in \mathscr{S}(\delta)$ and hence $\Delta_{\mathfrak{p}(X)} \in \mathscr{S}(H_s(\delta))$, making $H_s(\delta)$ discrete.

THEOREM 7. *Suppose (X,δ) is a p-space, $A \subseteqq X$. Then*

$$(\mathcal{P}(A), H_w(\delta|A)) = (\mathcal{P}(A), H_w(\delta)|\mathcal{P}(A)) .$$

PROOF. It is well known that

$$\big(\mathcal{P}(A), H(\mathscr{U}(\delta) \cap (A \times A))\big) = \big(\mathcal{P}(A), H(\mathscr{U}(\delta)) \cap (\mathcal{P}(A) \times \mathcal{P}(A))\big) .$$

Since these uniformities generate the proximities in question, the result follows.

Unfortunately a similar property does not hold for $H_s(\delta)$ as the next example illustrates.

EXAMPLE 1. Let (N,δ_M) be the natural numbers with the discrete proximity: let $(X,\mathscr{U})$ be the uniform completion of $(N,\mathscr{U}(\delta_M))$; and let $\delta=\delta(\mathscr{U})$. For each $n \in N$ let $A_n=\{m: m \geqq n\}$. Let

$$\mathcal{A} = \{A_{2n} : n \in N\} \quad \text{and} \quad \mathcal{B} = \{A_{2n-1} : n \in N\} .$$

Since $\mathcal{A} \cap \mathcal{B}=\emptyset$ and $H_s(\delta_M)$ is discrete, $\mathcal{A}$ is far from $\mathcal{B}$ in $H_s(\delta|N)=H_s(\delta_M)$. We will show that $\mathcal{A}$ is near $\mathcal{B}$ relative to $H_s(\delta)|\mathcal{P}(N)$. Since $(X,\mathscr{U})$ is compact, $\pi(\delta)=\{\mathscr{U}\}$, snd $\mathscr{U}=\mathscr{U}(\delta)$. Hence, given $U \in \mathscr{U}$ there is a finite proximal cover $\{D_i\}_{i=1}^n$ of X such that $\bigcup(D_i \times D_i) \subseteqq U$. Assume the indexing on the D_i's is such that $D_i \cap N$ is infinite for $1 \leqq i \leqq j^*$ and finite for $j^* < i \leqq n$. Let

$$m^* = \max\{n \in N : n \in D_i \text{ for some } i,\ j^* < i \leqq n\} + 1 .$$

We claim that

$$A_{2n} \in H(U)[\mathcal{B}] \cap \mathcal{A} \quad \text{for} \quad n > m^* .$$

We prove this by showing that

$$A_{m^*} \subseteqq U[A_m] \quad \text{for all} \quad m \geqq m^* .$$

Let $k \in A_{m^*}$. Then $k \in D_{i^*}$ for some i^*, $1 \leqq i^* \leqq j^*$. Now $A_m \cap D_{i^*} \neq \emptyset$, for if not, then $N \cap D_{i^*}$ would be contained in $\{n: 1 \leqq n < m\}$ and would thus be finite which it is not. Thus there is a $k^* \in A_m \cap D_{i^*}$. Then

$$(k^*, k) \in D_{i^*} \times D_{i^*} \subseteqq U \quad \text{and} \quad k \in U[k^*] \subseteqq U[A_m] .$$

Now if $n > m^*$, then $2n$ and $2n-1 > m^*$. Thus, since A_{2n-1}, A_{2n} are contained in A_{m^*}, it follows that

$$(A_{2n}, A_{2n-1}) \in H(U) \quad \text{and} \quad A_{2n} \in H(U)[\mathcal{B}] \cap \mathcal{A} .$$

Since $H(U)[\mathcal{B}] \cap \mathcal{A} \neq \emptyset$ for all $U \in \mathcal{U}$, $\mathcal{A}$ is near $\mathcal{B}$ relative to $H_s(\delta)$ and hence relative to $H_s(\delta)|\mathcal{P}(N)$.

The following lemma is related to the remarks made about $H_s(\delta)$ immediately following the definition. If $\mathcal{A}$ is a family of subsets of $X \times X$ and $f\colon X \to Y$, then

$$(f \times f)[\mathcal{A}] = \{(f \times f)[A] : A \in \mathcal{A}\} .$$

Lemma 3. *For any p-space* (X,δ),

$$\mathcal{S}(H_s(\delta)|i[X]) = \mathcal{S}(H_s(\delta)) \cap (i[X] \times i[X]) = (i \times i)[\mathcal{S}(\delta)] .$$

Proof. That $\mathcal{S}(H_s(\delta)|i[X]) = (i \times i)[\mathcal{S}(\delta)]$ follows from the remarks mentioned above. It is also obvious that

$$\mathcal{S}(H_s(\delta)) \cap (i[X] \times i[X]) \subseteq \mathcal{S}(H_s(\delta)|i[X]) .$$

Let $\mathcal{Q} \in \mathcal{S}(H_s(\delta)|i[X])$ and let $Q = (i \times i)^{-1}[\mathcal{Q}]$. Then by lemma 2, $H(Q) \in \mathcal{S}(H_s(\delta))$ and thus

$$H(Q) \cup \mathcal{Q} \in \mathcal{S}(H_s(\delta)) .$$

Since

$$(H(Q) \cup \mathcal{Q}) \cap (i[X] \times i[X]) = \mathcal{Q}$$

and is in $\mathcal{S}(H_s(\delta)) \cap (i[X] \times i[X])$, the lemma follows.

Various questions can be asked about the totality of one of $H_w(\delta)$, $H_s(\delta)$, or δ implying the totality of the others. Most of these questions are unanswered but we do have the following partial results.

Theorem 8. *If* $(\mathcal{P}(X), H_s(\delta))$ *is total, then* (X,δ) *is also total.*

Proof. It is sufficient to show that $\mathcal{S}(\delta)$ is a filter. Let U_1 and $U_2 \in \mathcal{S}(\delta)$. Since $H_s(\delta)$ is total and $H(\mathcal{S}(\delta)) \subseteq \mathcal{S}(H_s(\delta))$, there is a $\mathcal{Q} \in \mathcal{S}(H_s(\delta))$ such that

$$\mathcal{Q} \subseteq H(U_1) \cap H(U_2) .$$

By lemma 3 there is a $Q \in \mathcal{S}(\delta)$ such that

$$(i \times i)[Q] \subseteq \mathcal{Q} \cap (i[X] \times i[X]) .$$

It is easily shown that $Q \subseteq U_1 \cap U_2$, and therefore $\mathcal{S}(\delta)$ is a uniformity.

If $\mathcal{U}$ is a uniformity for X with a linearly ordered base, and if $X_0 \subseteq X$, then $\mathcal{U} \cap (X_0 \times X_0)$ also has a linearly ordered base. Therefore the theorem of [1] yields the following results.

THEOREM 9. *Suppose* (X,δ) *is a p-space.*

(1) *If* $\mathscr{S}(H_w(\delta))$ *has a linearly ordered base, then* (X,δ) *is total and* $\mathscr{S}(\delta)$ *has a linearly ordered base.*

(2) *If* $\mathscr{S}(\delta)$ *has a linearly ordered base, then* $(\mathfrak{P}(X), H_s(\delta))$ *is total and* $\mathscr{S}(H_s(\delta))$ *has a linearly ordered base.*

(3) *If* $\mathscr{S}(H_w(\delta))$ *has a linearly ordered base, then* $(\mathfrak{P}(X), H_s(\delta))$ *is total and* $\mathscr{S}(H_s(\delta))$ *has a linearly ordered base.*

In order to discuss completeness properties of $H_w(\delta)$ and $H_s(\delta)$ we briefly examine convergence of nets in hyperproximity spaces.

LEMMA 4. *Suppose* (X,δ) *is a p-space,* $S: D \to \mathfrak{P}(X)$ *is a net, and* $A \subseteq X$. *Then* $\lim S = A$ *relative to* $\mathscr{T}(H_s(\delta))$ *iff* S *is eventually in* $H(U)[A]$ *for all* $U \in \mathscr{S}(\delta)$.

PROOF. The lemma follows from the observation that

$$\begin{aligned}\mathscr{T}(H_s(\delta)) &= \sup\{\mathscr{T}\big(\delta(H(\mathscr{U}))\big) : \mathscr{U} \in \pi(\delta)\}\\ &= \sup\{\mathscr{T}(H(\mathscr{U})) : \mathscr{U} \in \pi(\delta)\}\,.\end{aligned}$$

LEMMA 5. *Suppose* (X,δ) *is a p-space and* $S: D \to \mathfrak{P}(X)$ *is a net. Suppose* $\lim S = A$ *relative to* $H_s(\delta)$ *(or* $H_w(\delta)$*). Then* $\lim S = cA$ *relative to* $H_s(\delta)$ *(or* $H_w(\delta)$*) where c denotes the closure operator in* $\mathscr{T}(\delta)$.

The proof of this result is routine and is left to the reader. The definition of lim sup for a sequence of sets can be found in Whyburn's work [12]. We use the concept here in the following form. If $S: D \to \mathfrak{P}(X)$ is a net and $(X, \mathscr{T})$ is a topological space, we define $L(S)$ to be the set of points x for which $x \in O \in \mathscr{T}$ implies that for each $d \in D$ there is a $d^* \in D$, $d^* \geqq d$, such that $S(d^*) \cap O \neq \emptyset$.

LEMMA 6. *Suppose* $S: D \to \mathfrak{P}(X)$ *and* (X,δ) *is a p-space. If S converges in* $\mathscr{T}(H_w(\delta))$ *or* $\mathscr{T}(H_s(\delta))$, *then S converges to* $L(S)$.

PROOF. We give the proof for $\mathscr{T}(H_s(\delta))$. Suppose S converges to A. If $A = \emptyset$, then S must be eventually $\emptyset$ since $H(U)[\emptyset] = \{\emptyset\}$. In this case $L(S) = \emptyset$. If $A \neq \emptyset$, we will show that $cA = L(S)$ and hence, by lemma 5, S converges to $L(S)$.

Let $x \in cA$ and let $U \in \mathscr{S}(\delta)$ be symmetric. We claim that, given $d \in D$ there is a $d^* \geqq d$ such that

$$S(d^*) \cap U[x] \neq \emptyset\,.$$

There is an $e^* \in D$, $e^* \geqq d$, such that $e \geqq e^*$ implies that $cA \subseteqq U[S(e)]$ by lemma 4. Hence

$$x \in U[S(e^*)] \quad \text{or} \quad U[x] \cap S(e^*) \neq \emptyset .$$

Thus $x \in L(S)$.

Suppose $x \in L(S)$ and $U \in \mathscr{S}(\delta)$. Let U^* be symmetric, $U^* \in \mathscr{S}(\delta)$, such that $U^* \circ U^* \subseteqq U$. There is an $e^* \in D$ such that $S(d) \subseteqq U^*[A]$ for all $d \geqq e^*$. There is a $d^* \geqq e^*$ such that

$$S(d^*) \cap U^*[x] \neq \emptyset .$$

Thus $x \in U^*[S(d^*)]$ which is contained in $U^*[U^*[A]] \subseteqq U[A]$. Hence $x \in cA$.

We note that Isbell has proved results similar to the above for hyperuniform spaces.

A p-space (X,δ) will be called *w-hypercomplete* if $(\mathfrak{P}(X), H_w(\delta))$ is complete and *s-hypercomplete* if $(\mathfrak{P}(X), H_s(\delta))$ is complete. The following relationship between the two types of hypercompleteness holds.

THEOREM 10. *If (X,δ) is w-hypercomplete, then it is s-hypercomplete.*

PROOF. Let $S\colon D \to \mathfrak{P}(X)$ be p-Cauchy in $H_s(\delta)$. Then, since $H_w(\delta) \leqq H_s(\delta)$,

$$\mathscr{S}(H_w(\delta)) \subseteqq S(H_s(\delta)) ,$$

and S is $H_w(\delta)$ p-Cauchy. Hence it converges to $L(S)$ in $\mathscr{T}(H_w(\delta))$. Let $U \in \mathscr{S}(\delta)$. Since $L(S) \Subset U[L(S)]$, there is a $V \in \mathscr{U}(\delta)$ such that $V[L(S)] \subseteqq U[L(S)]$. For V there is a $d' \in D$ such that $d \geqq d'$ implies

$$S(d) \subseteqq V[L(S)] \subseteqq U[L(S)] .$$

Let U^* be symmetric such that $U^* \in \mathscr{S}(\delta)$ and $U^* \circ U^* \subseteqq U$. Since S is $H_s(\delta)$ p-Cauchy there is a $d'' \in D$ such that $d_1, d_2 \geqq d''$ implies $(S(d_1), S(d_2)) \in H(U^*)$. Now suppose $x \in L(S)$. Then there is a $d = d(U^*) \geqq d''$ such that

$$U^*[x] \cap S(d) \neq \emptyset .$$

Then $x \in U^*[S(d)]$ which is contained in $U^*[U^*[S(e)]]$ for all $e \geqq d''$. Thus $L(S) \subseteqq U[S(e)]$ for all $e \geqq d''$. Choosing $d^* \geqq d'$ and d'' we have that for all $d \geqq d^*$,

$$(S(d), L(S)) \in H(U) ,$$

completing the proof.

That the two types of hypercompleteness are not equivalent will be shown later. The fact that an s-hypercomplete space, and hence a w-hypercomplete space, is complete will now be demonstrated.

THEOREM 11. *If* (X,δ) *is s-hypercomplete, then it is complete.*

PROOF. Let $S: D \to X$ be p-Cauchy. Then $i \circ S: D \to \mathcal{P}(X)$ is p-Cauchy in $(\mathcal{P}(X), H_s(\delta))$ and hence converges to some $A \subseteq X$. Since $i(S(d)) \neq \emptyset$ for any $d \in D$, $A \neq \emptyset$. It is easily shown that S converges to a for all $a \in A$.

It is well known that the hyperuniform space of a complete pseudometric space is complete (see [5]). From this, the fact that complete uniform spaces yield complete proximity spaces, and theorem 6 we deduce the following:

THEOREM 12. *If* (X,δ) *is a complete pseudometrizable p-space, then* (X,δ) *is s-hypercomplete.*

It is interesting to note that a similar theorem is not true for $H_w(\delta)$. As a matter of fact a discrete p-space need not be w-hypercomplete as is indicated by the following example.

EXAMPLE 2. Consider the space (N,δ_M). We claim that $(\mathcal{P}(N), H_w(\delta_M))$ is not p-complete. Let $S: N \to \mathcal{P}(N)$ be the sequence defined by

$$S(n) = \{m \in N : m \geqq n\}.$$

If $U \in \mathcal{U}(\delta_M)$, then there is a finite cover $\{D_i\}_{i=1}^{j}$ of N such that $\bigcup (D_i \times D_i) \subseteq U$. Defining m^* as in example 1 we can show that for m_1 and $m_2 \geqq m^*$,

$$(S(m_1), S(m_2)) \in H(U).$$

Thus S is Cauchy relative to $H(\mathcal{U}(\delta_M))$. Then by the corollary to lemma 1, S is p-Cauchy relative to $H_w(\delta_M)$.

Now if S converges it must converge to $L(S)$. But, since $\{x\}$ is a neighborhood of x for all $x \in N$, $\{x\} \cap S(n_k)$ must be non-empty for some subsequence $\{n_k\}$ if x is to be in $L(S)$. Since this is true for no $x \in N$, $L(S) = \emptyset$. Since $(S(n), \emptyset) \in H(U)$ for no $U \in \mathcal{U}(\delta_M)$, S does not converge.

Since, by theorem 12, (N, δ_M) is s-hypercomplete we have also shown that s-hypercompleteness and w-hypercompleteness are not equivalent.

In light of Isbell's work on uniform spaces some interesting questions about hypercompleteness arise which are only partially answered here. For example, it is not known if all complete p-spaces are s-hypercomplete. The relationship between paracompactness and s-hypercompleteness is not completely known, although the following result is clear in light of Isbell's work and theorem 9.

THEOREM 13. *Suppose* (X,δ) *is such that* $\mathscr{S}(\delta)$ *has a linearly ordered base. Then* (X,δ) *s-hypercomplete implies* $\mathscr{T}(\delta)$ *is paracompact.*

We close with the remark that for a compact p-space $H_w(\delta)$, and hence $H_s(\delta)$ by theorem 5, is compact. This follows from the fact that (X,δ) is compact iff $(X,\mathscr{U}(\delta))$ is compact, which will imply that $\big(\mathcal{P}(X), H(\mathscr{U}(\delta))\big)$ is compact.

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OAKLAND UNIVERSITY, ROCHESTER, MICH., U.S.A.