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REFLEXIVITY AND COMPLETENESS IN VECTOR-LATTICES

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In the present paper we are going to study vector-lattices which are Dedekind-complete, in conjunction with certain natural vector-topologies. No a priori topological assumptions are made, the topologies are all derived from the inherent vector and lattice properties of the underlying spaces.

Our main result, Theorem 2, states that if E is a Dedekind-complete vector-lattice, and E^c is the linear space of all order-continuous linear functionals on E, then the completion of E with respect to the Mackey-topology $\tau(E,E^c)$ is again a Dedekind-complete vector-lattice, and may be identified with a closed order-ideal of E^{cc}, the space of all order-continuous linear functionals on E^c.

1. Notation and basic concepts.

A vector-lattice E is *Dedekind-complete* if each majorized subset af E has a least upper bound. For all relevant information concerning such structures, we refer to [1] and [3], where also other references can be found. Here we are first going to give our notation, and a few facts that will be of use.

Throughout the paper, E,F will denote Dedekind-complete vector-lattices. The prefix "*o*-" to words in the text stands for "order-". A net $\{x_i\}_{i\in I}\subseteqq E$ is *o-convergent* to an element $x\in E$, and we write $x_i\to x$ (o) if there is a cofinal subset $I'\subseteqq I$ and two nets $\{y_{i'}\}_{i'\in I'}$, $\{z_{i'}\}_{i'\in I'}$ in E satisfying

(i) $y_{i'}\downarrow x$; $z_{i'}\uparrow x$; $i'\in I'$.
(ii) $z_{i'}\leqq x_i\leqq y_{i'}$ if $i\geqq i'$.

A linear functional f on E is *o-continuous* if $f(x_i)\to f(x)$ whenever $x_i\to x$ (o). We will always assume that the linear space E^c of all linear o-continuous functionals on E separates points of E.

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A subset $A \subseteq E$ is *normal* if $x \in A$ and $|y| \leqq |x| \Rightarrow y \in A$. A linear subspace $L \subseteq E$ is an *o-ideal* if L is a normal subset of E. An o-ideal L in E is a *direct factor* if it contains the supremum of any subset $A \subseteq L$ whenever the supremum exists in E. A linear functional on E is *o-bounded* if it is bounded on every o-interval $[x,y] = \{z \in E\colon x \leqq z \leqq y\}$ in E. The linear space of all o-bounded linear functionals on E is denoted E^b. The spaces E^c and E^b are Dedekind-complete vector-lattices under their natural ordering, and E^c is a direct factor of E^b (not necessarily distinct from E^b).

If $x \in E$ and L is a direct factor of E, then x_L denotes the component of x in L. For $f \in E^c$ let f_L be the linear functional $x \to f(x_L)$, then $f_L \in E^c$. A Hahn-decomposition is possible: if $f \in E^c$, then E is the direct sum of three direct factors: $E = Z \oplus P \oplus N$ such that f vanishes identically on $Z =$ the *zero-ideal* of f, and is strictly positive (negative) on P (N), the *positive* (*negative*) *ideal* of f. Explicitly, we have

$$P = \{x \in E\colon 0 < y \leqq |x| \Rightarrow f(y) > 0\},$$

and similarly for Z and N.

Each element $x \in E$ determines an o-continuous linear functional on E^c by the canonical map of E into the algebraic dual of E^c. In this way we may regard E as an o-ideal of E^{cc}. The canonical injection of E into E^{cc} is o-continuous, if $x_i \to x$ (o) in E, then $x_i \to x$ (o) also in E^{cc}. E is *o-reflexive* if $E = E^{cc}$.

If E and F are in duality, $\sigma(E,F)$ denotes the weak topology of E with respect to F, and $\tau(E,F)$ denotes the Mackey-topology of E with respect to F. If $\mathscr{S}$ is a family of subsets of F, $\tau_{\mathscr{S}}$ denotes the topology for E of uniform convergence on the sets in $\mathscr{S}$. The topology $\tau_{\mathscr{S}}$ is a (locally convex) vector topology for E if and only if each set in $\mathscr{S}$ is $\sigma(F,E)$-bounded. For this and other statements of purely linear topological nature, we refer to [2].

A vector-topology for E is *o-continuous* if o-convergence implies convergence with respect to the topology. The topology $\tau(E,E^c)$ is the strongest o-continuous locally convex topology for E, thus a locally convex topology for E is o-continuous if and only if it is weaker than $\tau(E,E^c)$. A vector-topology for E is *normal* if it has a local base of normal subsets. The lattice-operations are continuous with respect to a normal topology, and $\tau(E,E^c)$ is normal.

2. The main results.

We consider E as an o-ideal of E^{cc}. Our first result states that there is a normal, locally convex, o-continuous topology $\tau_{\mathscr{B}}$ for E, compatible

with the duality (E, E^c) such that the completion of E with respect to $\tau_{\mathscr{B}}$ may be identified with E^{cc}.

Let $\mathscr{B}$ be the family of $\sigma(E^c, E^{cc})$-compact, convex, subsets of E^c, and let $\mathscr{A}$ be the family of $\sigma(E^c, E)$-compact, convex subsets of E^c. Since $E \subseteq E^{cc}$, we have $\mathscr{B} \subseteq \mathscr{A}$. We give E^{cc} the topology $\tau_{\mathscr{B}}$. The topology $\tau_{\mathscr{B}}$ restricted to E is clearly weaker than $\tau_{\mathscr{A}} = \tau(E, E^c)$ on E. Hence the dual of E with respect to $\tau_{\mathscr{B}}$ is equal to E^c.

THEOREM 1. *The topology $\tau_{\mathscr{B}}$ is normal, and is the strongest o-continuous locally convex topology on E^{cc}. With this topology E^{cc} is complete, and contains E as a dense subspace.*

An immediate consequence of this result is that the completion of E with respect to $\tau_{\mathscr{B}}$ is again a Dedekind-complete vector-lattice, containing E as a dense o-ideal. Of greater interest however, is to know that this is also true for the completion of E with respect to the Mackey-topology $\tau_{\mathscr{A}}$.

THEOREM 2. *The topology $\tau_{\mathscr{A}}$ is a normal, locally convex topology for E^{cc}, relative to which E^{cc} is complete. The closure $\bar{E}$ of E in E^{cc} with respect to $\tau_{\mathscr{A}}$ is a Dedekind-complete vector-lattice and an o-ideal of E^{cc}. $\tau_{\mathscr{A}}$ is the strongest o-continuous locally convex topology for $\bar{E}$, and $\bar{E} = E^{cc}$ if and only if $\tau_{\mathscr{A}}$ is o-continuous on E^{cc}.*

The following example is now quite pertinent. In the context of Theorem 2, it shows that neither E nor $\bar{E}$ *need be equal to* E^{cc}.

Let c_0 be the set of all real sequences $\{a_n\}$ such that $a_n \to 0$ as $n \to \infty$, and let l^∞ be the set of all bounded real sequences. c_0 and l^∞ are Dedekind-complete vector-lattices under their natural ordering. We give l^∞ the norm-topology

$$\|a\| = \sup_n |a_n|; \qquad a = \{a_n\} \in l^\infty .$$

Then $c_0 = \bar{c}_0 \subseteq l^\infty$. The norm-dual of c_0 is linearly isometric to l^1, and one may show that convergence in norm is the same as o-convergence in c_0. Hence $(c_0)^c = l^1$. It follows that the family of closed balls in l^1 is a co-base for $\mathscr{A}$ = the family of $\sigma(l^1, c_0)$-compact, convex subsets of l^1. Moreover, the norm-dual of l^1 is linearly isometric to l^∞, and we have $(l^1)^c = l^\infty$. Hence $\tau_{\mathscr{A}}$ is the norm-topology for l^∞, and $l^\infty = (c_0)^{cc}$. Note in particular that the norm-topology is not o-continuous on l^∞.

3. Proofs and related results.

Again, we consider E as an o-ideal in E^{cc}.

LEMMA 1.

(a) *E^{cc} is the least direct factor containing E;*

(b) $E^c = E^{ccc}$.

PROOF. Let L be the complementary direct factor of E in E^{cc}, and take $X \geqq 0$ in L. For any $x \in E$, $x \geqq 0$, we have $x \wedge X = 0$. Let P be the positive ideal of X in E^c, and let Z_x, P_x be the zero-ideal and positive ideal of x in E^c, respectively. We claim that $P_x \cap P = (0)$. Indeed, if $f \in E^c$, $f > 0$, we have

$$(x \wedge X)(f) = \inf \{x(f-g) + X(g) : 0 \leqq g \leqq f\} .$$

The function $g \to x(f-g) + X(g)$ is $\sigma(E^c, E^{cc})$-continuous on E^c, the o-interval $[0,f]$ is $\sigma(E^c, E^{cc})$-compact [1], so this function will obtain its minimal value on $[0,f]$. If f belongs to $P_x \cap P$, however, then $x(f-g) + X(g) > 0$ for all $g \in [0,f]$. Hence $(x \wedge X)(f) > 0$, a contradiction, so $P_x \cap P = (0)$. Now $E^c = Z_x \oplus P_x$, so we get $P \subseteqq Z_x$. Since x was arbitrary, it follows that

$$P \subseteqq \bigcap \{Z_x : x \in E, x \geqq 0\} = (0) .$$

Hence $X = 0$, and consequently $L = (0)$, proving (a).

To prove (b), let r be the restriction map $f \to f|E$; $f \in E^{ccc}$. Then r maps E^{ccc} onto E^c, for if i: $E^c \to E^{ccc}$ is the evaluation map, then $r \circ i$ is the identity map on E^c. r is injective, for if $r(f) = f|E = 0$, then $f = 0$ by (a). Hence $E^c = E^{ccc}$, and Lemma 1 is proved.

LEMMA 2. *Let F be a Dedekind-complete vector-lattice. Then F^c is complete in the topology $\tau(F^c, F)$.*

PROOF. By the Grothendieck completeness theorem it is sufficient to prove that if f is a linear functional on F which is $\sigma(F, F^c)$-continuous on each convex, $\sigma(F, F^c)$-compact subset of F, then f belongs to F^c. So let f be such a functional, and suppose that $x_i \to x$ (o) in F. Then $x_i \to x$ in the $\sigma(F, F^c)$-topology, and there exist elements $y, z \in F$ such that $i \geqq i_0$ implies $y \leqq x_i \leqq z$. The o-interval $[y,z]$ is convex and $\sigma(F, F^c)$-compact, so $f|[y,z]$ is $\sigma(F, F^c)$-continuous. It follows that $f(x_i) \to f(x)$, so $f \in F^c$.

PROOF OF THEOREM 1. Since $\tau_{\mathscr{B}} = \tau(E^{cc}, E^c)$, we obtain by Lemma 2, taking $F = E^c$, that E^{cc} is complete with respect to $\tau_{\mathscr{B}}$. By Lemma 1 (b),

we conclude that $E^c = E^{ccc}$, so $\tau_{\mathscr{B}} = \tau(E^{cc}, E^{ccc})$, and this topology is normal and the strongest o-continuous locally convex topology for E^{cc}. By Lemma 1(a) it also follows that if $X \geqq 0$ is any element in E^{cc}, then there exists a monotone net $\{x_i\}_{i \in I} \subseteq E$ such that $x_i \uparrow X$ in E^{cc}. Since $\tau_{\mathscr{B}}$ is o-continuous, it follows that E is dense in E^{cc}.

Recall that $\mathscr{A}$ is the family of convex, $\sigma(E^c, E)$-compact subsets of E^c. It is not a priori clear that $\tau_{\mathscr{A}}$ is a vector-topology for E^{cc}, so we will first establish this. Let $s = s(E^c, E)$ be the strong topology for E^c, i.e. the topology of uniform convergence on $\sigma(E, E^c)$-bounded subsets of E, and let E^{c*} denote the dual of E^c in this topology.

LEMMA 3. $E^{c*} \supseteqq E^{cc}$.

PROOF. It is sufficient to show that s is stronger than $\sigma(E^c, E^{cc})$, for the dual of E^c with respect to the latter is exactly E^{cc}. Let $X \in E^{cc}$, $X \geqq 0$ and let

$$V = \{f \in E^c : |X(f)| \leqq 1\}.$$

Then V is a $\sigma(E^c, E^{cc})$-neighbourhood of 0 in E^c, and any other $\sigma(E^c, E^{cc})$-neighbourhood of 0 will contain a set V of this type. Let $A = \{x \in E : 0 \leqq x \leqq X\}$; then A is a $\sigma(E, E^c)$-bounded subset of E. Indeed, if $f \in E^c$ and $x \in A$, then $|f(x)| \leqq X(|f|)$, that is

$$\sup_{x \in A} |f(x)| < \infty .$$

Hence $U = \{f \in E^c : |f(x)| \leqq 1, \forall x \in A\} = A^0$ is a neighbourhood of 0 in E^c in the s-topology. Now there is an increasing net $\{x_i\} \subseteq A$ such that $x_i \uparrow X$ in E^{cc}. Hence, if $f \in U$ we get

$$|X(f)| = \lim_i |f(x_i)| \leqq 1$$

since f is o-continuous on E^{cc}. It follows that $U \subseteq V$ which proves the Lemma.

COROLLARY. *If $A \in \mathscr{A}$, then A is $\sigma(E^c, E^{cc})$-bounded.*

PROOF. Since A is $\sigma(E^c, E)$-compact, A is strongly bounded, and hence it is $\sigma(E^c, E^{c*})$-bounded. By the Lemma it then follows that A is $\sigma(E^c, E^{cc})$-bounded.

PROOF OF THEOREM 2. The corollary states that $\tau_{\mathscr{A}}$ is a locally convex (vector) topology for E^{cc}. Since the closed normal hull of a set $A \in \mathscr{A}$ is in $\mathscr{A}$ [1], we may assume the sets in $\mathscr{A}$ to be normal, and $\tau_{\mathscr{A}}$ is therefore

normal. Since $\tau_{\mathcal{A}}$ is stronger than $\tau_{\mathcal{B}}$, and both are admissible topologies for E^{cc} relative E^c, we know [2, 18.3] that E^{cc} is complete with respect to $\tau_{\mathcal{A}}$.

Let $\bar{E}$ be the closure of E in E^{cc} with respect to $\tau_{\mathcal{A}}$. Since E^{cc} is Dedekind-complete, it will follow that $\bar{E}$ is Dedekind-complete once we have proved that $\bar{E}$ is an o-ideal of E^{cc}. $\bar{E}$ is a linear subspace of E^{cc}, so it is sufficient to prove

(i) if $X \in \bar{E}$, then $X^+ \in \bar{E}$,
(ii) if $0 \leqq Y \leqq X \in \bar{E}$, $Y \in E^{cc}$, then $Y \in \bar{E}$.

Here (i) is easily settled: let $X \in \bar{E}$ and let $\{x_i\}_{i \in I}$ be a net in E satisfying $x_i \to X$ with respect to $\tau_{\mathcal{A}}$. Since $\tau_{\mathcal{A}}$ is normal, $x_i^+ \to X^+$, and $\{x_i^+\}_{i \in I} \subseteq E$, so $X^+ \in \bar{E}$.

To verify (ii), we first prove the following: if $X \in \bar{E}$, $X \geqq 0$, and $\{L_i\}_{i \in I}$ is a monotone, descending family of direct factors in E^c such that $\bigcap_{i \in I} L_i = (0)$, then $X_{L_i} \to 0$ with respect to $\tau_{\mathcal{A}}$. Indeed, let $A \in \mathcal{A}$ and $e > 0$ be given, and choose $x \in E$ such that

$$|X(f) - x(f)| < \tfrac{1}{2}e \quad \text{for all} \quad f \in A .$$

Now $x_{L_i} \in E$ for all $i \in I$ since E is an o-ideal of E^{cc}, $x_{L_i} \to 0\,(o)$, and $\tau_{\mathcal{A}}$ is o-continuous on E, so there is an $i_0 \in I$ such that $i \geqq i_0$ implies $|x_{L_i}(f)| < \frac{1}{2}e$ for all $f \in A$. Hence, if $i \geqq i_0$ and $f \in A$,

$$|X_{L_i}(f)| = |X(f_{L_i})| \leqq |X(f_{L_i}) - x(f_{L_i})| + |x(f_{L_i})| < \tfrac{1}{2}e + \tfrac{1}{2}e = e ,$$

since $f \in A \Rightarrow f_L \in A$ for any direct factor $L \subseteq E^c$, since A is normal. It follows that $X_{L_i} \to 0$ with respect to $\tau_{\mathcal{A}}$. Now let $0 \leqq Y \leqq X \in \bar{E}$, $Y \in E^{cc}$. The space $\bar{E}$ is the linear space of those linear functionals on E^c which are $\sigma(E^c, E)$-continuous on the sets $A \in \mathcal{A}$, so we will show that Y has this property. We first claim that there is a maximal direct factor $L \subseteq E^c$ such that $Y_L \in \bar{E}$. If we then can show that $L = E^c$, we will be finished. Suppose that $\{L_i\}_{i \in I}$ is any totally ordered (by inclusion) upward directed family of direct factors in E^c such that $Y_{L_i} \in \bar{E}$ for each $i \in I$. Let $L = \operatorname{lub}_{i \in I} L_i$. Put $M_i = L \cap L_i'$, where L_i' is the complementary direct factor of L_i in E^c. Then $\bigcap_{i \in I} M_i = (0)$, so by the result proved above $X_{M_i} \to 0$ uniformly on each $A \in \mathcal{A}$. Pick any $A \in \mathcal{A}$; since it is normal we get for $f \in A$

$$|Y_{M_i}(f)| \leqq |Y_{M_i}(|f|)| \leqq |X_{M_i}(|f|)| \to 0$$

uniformly, so also $Y_{M_i} \to 0$ uniformly on A. It follows that $Y_{Li} \to Y_L$ uniformly on A, so by the assumption on Y_{L_i} we obtain that $Y_L \in \bar{E}$. By Zorn's Lemma this implies the existence of a maximal L such that

$Y_L \in \bar{E}$. Suppose $L \neq E$; then $Y_{L'} > 0$ so there is an element $f > 0$, $f \in L'$, such that $Y_{L'}(f) = Y(f) > 0$. The positive cone in E^c is convex and $\sigma(E^c, E)$-closed, so there is an element $x > 0$ in E such that $0 < Y(f) < f(x)$. Let P be the positive ideal of $x - Y$, and put $L_1 = P \cap L'$. Since $f \in L'$, $L_1 \neq (0)$. It follows that $Y_{L_1} \leqq x$, so $Y_{L_1} \in E$ since E is an o-ideal in E^{cc}. Hence

$$Y_{L_1 \oplus L} = Y_{L_1} + Y_L \in \bar{E},$$

contradicting the maximality of L. Hence $L = E^c$ and $Y \in \bar{E}$, so (ii) is proved. Thus we know that $\bar{E}$ is Dedekind-complete and an o-ideal in E^{cc}.

The topology τ_a restricted to $\bar{E}$ is the Mackey-topology $\tau(\bar{E}, E^c)$. Clearly $(\bar{E})^c = E^c$, so τ_a is o-continuous on $\bar{E}$. It follows that if $\bar{E} = E^{cc}$, then τ_a is o-continuous on E^{cc}. On the other hand, if τ_a is o-continuous on E^{cc}, then τ_a and $\tau_{\mathscr{B}}$ must coincide on E^{cc}, since by Theorem 1 we know that $\tau_{\mathscr{B}}$ is the strongest locally convex o-continuous topology on E^{cc}, and that $\tau_{\mathscr{B}}$ is weaker than τ_a. By Theorem 1, E is therefore dense in E^{cc} with respect to τ_a.

This proves Theorem 2.

A lot more can be said about the space E^c with its strong topology s. We collect some of the information, most of which is well known, in the following proposition:

PROPOSITION 1. *Let E^c have the strong topology $s = s(E^c, E)$. Then*

(i) *s is normal and complete,*
(ii) $E^{cc} \subseteqq E^{c*} \subseteqq E^{cb}$,
(iii) *$E^{cc} = E^{c*}$ if and only if s is o-continuous,*
(iv) *$E^{c*} = E^{cb}$ if s is bound.*

PROOF. (i) s is the topology of uniform convergence on the τ_a-bounded subsets of E. Since τ_a is normal, it follows easily that the normal hull of each τ_a-bounded set is bounded, so s is normal. s is clearly stronger than $\tau(E^c, E)$, so E^c is complete with respect to s since it is complete with respect to $\tau(E^c, E)$ (Lemma 2).

(ii) We already know that $E^{cc} \subseteqq E^{c*}$ (Lemma 3). Since s is normal, it follows readily that each o-interval of E^c is strongly bounded. Hence $E^{c*} \subseteqq E^{cb}$.

(iii) The topology s is o-continuous if and only if s is weaker than $\tau(E^c, E^{cc})$, which is the case if and only if $E^{c*} \subseteqq E^{cc}$. Combined with (ii), this gives (iii).

(iv) Follows from (A4) and (A5) in [1].

Still more can be said about E^c with strong topology; we refer to [1]. The statements above cover what we shall need in the next section.

4. Reflexivity.

We have already seen that E^c is o-reflexive (Lemma 1(b)), and it follows immediately that also E^{cc} is o-reflexive, while E itself need not be. In this section we look into the relationship between o-reflexivity and topological reflexivity for E.

PROPOSITION 2. *The following are equivalent*:

(i) *E is o-reflexive,*
(ii) *E is $\tau_{\mathscr{B}}$-complete,*
(iii) *E is τ_a-complete, and τ_a is o-continuous on E^{cc}.*

PROOF. Immediate from Theorems 1 and 2.

PROPOSITION 3. *The following are equivalent*:

(i) *E is semi-reflexive (rel. E^c),*
(ii) *E is o-reflexive and s is o-continuous on E^c.*

PROOF. We have $E \subseteq E^{cc} \subseteq E^{c*}$. If E is semi-reflexive, then $E = E^{cc} = E^{c*}$ so E is o-reflexive and s is o-continuous by (iii) of Proposition 1. Conversely (ii) implies $E = E^{cc}$ and $E^{cc} = E^{c*}$ by (iii) of Proposition 1, so E is semi-reflexive.

NOTE. The topology s need not be o-continuous even if E is o-reflexive. From the example following Theorem 2 it is clear that l^1 is o-reflexive, but the strong topology for $(l^1)^c = l^\infty$ is the norm-topology, which is not o-continuous.

PROPOSITION 4. *For the statements below, generally* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

(i) *$E = E^{bb}$,*
(ii) *E is o-reflexive, s is o-continuous and $E^c = E^b$,*
(iii) *E is reflexive with respect to τ_a.*

If E is metrizable with respect to τ_a, then all three statements are equivalent.

PROOF. (i) $\Rightarrow$ (ii): The space E^c is a direct factor of E^b, so there is an injection of E^{cc} in $E^{bc} \subseteq E^{bb}$. Hence $E = E^{bb}$ implies $E = E^{cc}$. If $E^c \neq E^b$, then there exists an element $X \neq 0$ in $E^{bb} = E$ vanishing identically

on E^c, a contradiction. We have $E^{cc} \subseteqq E^{c*} \subseteqq E^{cb} = E^{cc}$, so by Proposition 1 the topology s is o-continuous.

(ii) $\Rightarrow$ (iii): By Proposition 3 the vector-lattice E is semi-reflexive, and $E^c = E^b$ implies that $\tau_a = \tau(E, E^b)$ which is bound [1, (A4)], so E is evaluable. Hence E is reflexive.

If E is metrizable, then it is bound [2, 22.3]. We prove the implication (iii) $\Rightarrow$ (i). By reflexivity, E is o-reflexive (Proposition 3) and hence complete with respect to τ_a (Proposition 2), so $E^c = E^b$ since τ_a is bound [1, (A5)]. Moreover, s is bound since E is metrizable and reflexive [2, 22.16], so from Proposition 1 it follows that $E^{c*} = E^{cb}$, and consequently $E = E^{bb}$.

Recall that a positive element $e \in E$ is a *strong o-unit* for E if for any $x \in E$ there is a positive integer n such that $x \leqq ne$. The last part of this paper is devoted to a proof of the fact that if E has a strong o-unit, then E is o-reflexive. This is, at least indirectly, a well known result. A proof may be obtained using representation theory and theorems from measure theory. Indeed, if E has a strong o-unit it may be represented as an abelian von Neumann algebra, and from this fact one may go on to prove the o-reflexivity. However, our approach is completely intrinsic, and uses order and linear topological space methods only. Of course it yields an alternative proof of the fact that each abelian von Neumann algebra is a dual space as a Banach-space (which is true even in the non-abelian case).

PROPOSITION 5. *If E has a strong o-unit, then E is o-reflexive.*

PROOF. We prove the stronger result (Proposition 3) that E is semi-reflexive. We first make E into a normed linear space; for $x \geqq 0$, $x \in E$, let

$$\|x\| = \inf\{\lambda \in R : \lambda e \geqq x\} .$$

Then, for a general x in E, let

$$\|x\| = \max\{\|x^+\|, \|x^-\|\} .$$

It is well known that since E is archimedean it becomes a normed linear space this way. The unit sphere S in E is given by

$$S = \{x \in E : -e \leqq x \leqq e\}.$$

Since S is an o-interval, it is $\sigma(E, E^c)$-compact, so if we can show that each $\sigma(E, E^c)$-bounded set is norm-bounded, we will be finished.

Let E^* be the norm-dual of E. Since each element $f \in E^c$ is o-bounded,

we have $E^c \subseteq E^*$, and it is easily seen that E^c is a norm-closed linear subspace of E^*. Let S^* be the unit ball of E^*. We claim that if $x \in E$, then

$$\|x\| = \sup \{|f(x)| : f \in E^c \cap S^*\}.$$

Note first that if $\Omega = \{f \in E^c : f(e) = 1;\ f \geqq 0\}$, then $f(x) \geqq 0$ for all $f \in \Omega$ implies $x \geqq 0$ if $x \in E$. Indeed, if $x^- \neq 0$, there is $f \in \Omega$ satisfying $f(x^-) > 0$. Let L be the direct factor generated by x^-. Then $f_L \neq 0$, and $g = (f_L(x))^{-1} f_L \in \Omega$, and $g(x) = -g(x^-) < 0$. Now let

$$\Omega_0 = \{x \in E :\ |f(x)| \leqq 1;\ \forall f \in \Omega\}.$$

By the argument above, $x \in \Omega_0$ if and only if $x \in S$, and consequently

$$\Omega_0{}^0 = \{f \in E^* :\ |f(x)| \leqq 1;\ \forall x \in \Omega_0\} = S^*.$$

Now, if $x \in E$, we have $\|x\| = \sup_{f \in S^*} |f(x)|$, and since S^* is the $\sigma(E^*, E)$-closed, circled, convex hull of $\Omega \subseteq E^c \cap S^*$, the claim is proved. But then it follows [4, Theorem 4.4B], that a $\sigma(E, E^c)$-bounded subset of E is norm-bounded. The proof is complete.

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