IMMERSIONS OF COMPLEX FLAGMANIFOLDS

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In the following $G$ denotes a compact connected Lie group. If $H$ is any subgroup of $G$, we denote by $C(H)$ the centralizer of $H$ in $G$. Moreover if $X_0$ is a non-zero element in the Lie algebra $\mathcal{L}(G)$ of $G$, and $L$ is the one-parameter subgroup of $G$ in the direction $X_0$, we put $C(X_0) = C(L)$. The purpose of this paper is to prove the following immersion theorem, where $T$ is a maximal torus in $G$ such that $X_0 \in \mathcal{L}(T)$:

**THEOREM 1.** Let $X_0 \in \mathcal{L}(T) \backslash \{0\}$ belong to some singular hyperplane in $\mathcal{L}(T)$, and define $S$ to be the intersection of the singular hyperplanes to which $X_0$ belongs. Then $G/C(X_0)$ can be immersed in $\mathbb{R}^{n-p}$, where $n = \dim G$ and $p = \dim S$.

An explanation of the terminology follows below. In case $G = U(n)$, Theorem 1 gives results on immersions of complex flagmanifolds (see Theorem 2 below).

1.

It is well known that the subgroups $C(X_0)$ are connected (they are precisely the centralizers of tori in $G$), and that the adjoint representation gives an embedding

$$G/C(X_0) \overset{\varphi}{\to} \mathcal{L}(G),$$

defined by

$$\varphi(gC(X_0)) = (\text{Ad}g)X_0.$$

In this way $G/C(X_0)$ becomes a submanifold of $\mathcal{L}(G)$ whose tangent space at the point $(\text{Ad}g)X_0$ is

$$\{[X, (\text{Ad}g)X_0] \mid X \in \mathcal{L}(G)\}.$$

Now equip $\mathcal{L}(G)$ with an inner product $(\cdot, \cdot)$ such that $\text{Ad}g$ is an isometry of $\mathcal{L}(G)$ for every $g \in G$. Then the following relation holds for all $X, Y, Z$ in $\mathcal{L}(G)$:

$$X \cdot [Y, Z] = ([X, Y], Z).$$

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We choose an orthonormal basis in $\mathcal{L}(G)$ such that $\mathrm{Ad} | T$ with respect to this basis is described by the matrix

\[
\begin{pmatrix}
\cos 2\pi \theta_1(X) & -\sin 2\pi \theta_1(X) \\
\sin 2\pi \theta_1(X) & \cos 2\pi \theta_1(X) \\
\vdots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\exp X \rightarrow
\begin{pmatrix}
\cos 2\pi \theta_m(X) & -\sin 2\pi \theta_m(X) \\
\sin 2\pi \theta_m(X) & \cos 2\pi \theta_m(X) \\
\vdots & \ddots & \ddots & \ddots \\
1 & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

for $X \in \mathcal{L}(T)$. Here $\theta_\nu$ are non-zero elements of $\mathrm{Hom}_R(\mathcal{L}(T), R)$ which map $\mathcal{L}(T) \cap \exp^{-1}(0)$ into $\mathbb{Z}$, and the number of 1's is $l = \dim T$. We put

\[
S_\nu = \{ X \in \mathcal{L}(T) \mid \theta_\nu(X) = 0 \}.
\]

$S_1, S_2, \ldots, S_m$ are the so called singular hyperplanes in $\mathcal{L}(T)$. These are known to be mutually different (cf. Hopf [4]) and the Weyl group $W$ as a finite group of isometries of $\mathcal{L}(T)$ is generated by the reflections in $S_1, S_2, \ldots, S_m$. The analytic subgroup $U_\nu$ of $T$ with Lie algebra $S_\nu$ is an $(l-1)$-dimensional torus in $T$ (Hopf [4]).

Suppose that $X_0$ lies in $k$ of the singular hyperplanes, say $S_{r_1}, S_{r_2}, \ldots, S_{r_k}$, and put

\[
S = \bigcap_{i=1}^k S_{r_i}, \quad U = \bigcap_{i=1}^k U_{r_i}.
\]

If $k = 0$ it is understood that $S = \mathcal{L}(T)$ and $U = T$.

**Lemma.** With the above notation we have $C(X_0) = C(U)$.

**Proof.** The one-parameter subgroup $L$ of $T$ in the direction $X_0$ is contained in $U$, and therefore

\[
C(U) \subseteq C(L) = C(X_0).
\]

Since $C(X_0)$ is connected, we only have to prove that

\[
\dim C(U) \geq \dim C(X_0).
\]

That $X_0$ lies in exactly $k$ singular hyperplanes implies that

\[
\dim C(X_0) = l + 2k.
\]
From the definition of \( U \) and (3) one concludes that
\[
\dim C(U) \geq l + 2k.
\]

**Proposition.** The normal bundle of \( G/C(X_0) \) in \( \mathcal{L}(G) \) has \( p \) linearly independent cross sections, where \( p = \dim S \).

**Proof.** We will define a bundle homomorphism
\[
G/C(X_0) \times S \xrightarrow{\varphi} \hat{\nu},
\]
where \( \hat{\nu} \) is the normal bundle of \( G/C(X_0) \) in \( \mathcal{L}(G) \), by
\[
\varphi((\text{Ad}g)X_0, Y) = ((\text{Ad}g)X_0, (\text{Ad}g)Y).
\]
To see that \( \varphi \) is well defined, we first prove that, if \( h \in C(X_0) \), then \( (\text{Ad}h)Y = Y \), but this is true since \( h \in C(U) \) and \( Y \in \mathcal{L}(U) \). Secondly we prove that \( (\text{Ad}g)Y \) is orthogonal to the tangent space (1). This follows from (2):
\[
[[X, (\text{Ad}g)X_0], (\text{Ad}g)Y] = (X, [\text{Ad}g]X_0, (\text{Ad}g)Y)] = (X, (\text{Ad}g)[X_0, Y]) = 0,
\]
since \( X_0 \) and \( Y \) belong to the abelian Lie algebra \( \mathcal{L}(T) \). Obviously \( \varphi \) defines \( p \) linearly independent cross sections of \( \hat{\nu} \).

As a corollary we have the following theorem of Borel–Hirzebruch [1].

**Corollary.** \( G/T \) is a \( \pi \)-manifold.

**Proof.** If \( X_0 \notin S_\nu \) for all \( \nu \), we have \( C(X_0) = T \) and \( S = \mathcal{L}(T) \), so \( \varphi \) becomes a trivialization of \( \hat{\nu} \).

Now we can prove Theorem 1. The main tool is the following theorem of Hirsch [3]:

**Theorem (Hirsch).** Let \( M^n \) be a \( C^\infty \)-manifold of dimension \( n \) and \( \tau(M^n) \) its tangent bundle. If \( \eta \) is a real \( k \)-dimensional vector bundle \( (k \geq 1) \) over \( M^n \) such that \( \tau(M^n) \oplus \eta \) is trivial, then \( M^n \) can be immersed in euclidean space \( \mathbb{R}^{n+k} \) of dimension \( n + k \).

Hirsch also proved, that the immersion can be chosen to have \( \eta \) as normal bundle, but the above will be sufficient.

**Proof of Theorem 1.** If \( X_0 \) belongs to at least one singular hyper-plane we have a positive dimensional subbundle \( \eta \) of \( \hat{\nu} \) complementary
to the trivial $p$-dimensional subbundle given by the proposition. Thus

$$\tau \oplus \eta \oplus p = n$$

from which one can conclude that

$$\tau \oplus \eta = n - p.$$  

Now we apply the theorem of Hirsch to get Theorem 1.

The above theorem of Borel–Hirzebruch and the theorem of Hirsch show that $G/T$ can be immersed in $\mathbb{R}^{k+1}$, where $k = \dim(G/T)$.

If $X_0$ belongs to only one singular hyperplane, $C(X_0)$ has dimension $l+2$ and $S$ dimension $l-1$. In this case Theorem 1 shows that $G/C(X_0)$ is immersible in euclidean space with codimension 3.

2. Now we shall apply Theorem 1 in the case $G = U(n)$. As the maximal torus $T$ we take the diagonal matrices and $\mathcal{L}(T)$ is identified with $\mathbb{R}^n$ in the obvious way. The singular hyperplanes in $\mathbb{R}^n$ are then given by

$$S_{ij} = \{(x_1, \ldots, x_n) \mid x_i = x_j\}, \quad i > j.$$  

Let $X_0 \in \mathbb{R}^n$ have the first $n_1$ coordinates equal to $y_1$, the next $n_2$ coordinates equal to $y_2$ and so on. The last $n_q$ coordinates are then equal to $y_q$, $\sum n_i = n$, and we assume that $y_1, \ldots, y_i, \ldots, y_q$ are mutually distinct. Then an easy matrix calculation shows that

$$C(X_0) = U(n_1) \times U(n_2) \times \ldots \times U(n_q).$$

Moreover $S$ is the subspace of $\mathbb{R}^n$ given by the condition, that the first $n_1$ coordinates are equal, the next $n_2$ coordinates are equal and so on. Since the dimension of $S$ is $q$, we get

**Theorem 2. The complex flagmanifold**

$$W(n; n_1, \ldots, n_q) = U(n)/U(n_1) \times U(n_2) \times \ldots \times U(n_q), \quad n = \sum n_i,$$

where $q \leq n - 1$, can be immersed in $\mathbb{R}^{n^2 - q}$.

If $q = n$ it is a $\pi$-manifold and can then be immersed with codimension 1.

The flagmanifold $W(n; n_1, \ldots, n_q)$ has dimension

$$n^2 - \sum_{i=1}^{q} n_i^2,$$

and the theorem gives an immersion in euclidean space of codimension
\[ \sum_{i=1}^{q} (n_i^2 - 1) . \]

Notice that the manifolds
\[ W(n+k; n_1, \ldots, n_q, 1, 1, \ldots, 1), \quad k \text{ numbers } 1, \]
are immersible in euclidean space with a codimension independent of \( k \), whereas the dimensions of the manifolds tends to infinity as \( k \to \infty \).

REFERENCES

2. R. Bott, Morse theory and its application to homotopy theory, (Mimeographed, Notes by A. van de Ven), Bonn, 1960.

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