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RAMANUJAN'S SUM AND NAGELL'S TOTIENT FUNCTION FOR ARITHMETICAL SEMI-GROUPS

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1. Introduction.

The arithmetical semi-group for this work was first defined by Arne Beurling in [1]; here it is defined as follows. Suppose there is given a finite or infinite sequence $\{p\}$ of real numbers, called generalised primes, such that $0 < 1 < p_1 < p_2 < \ldots$. Form the set $\{l\}$ of all p-products, that is, products $p_1^{v_1} p_2^{v_2} \ldots$, where $v_1, v_2, \ldots$, are integers $\geqq 0$ of which all but a finite number are 0. Call these numbers generalised integers and suppose that the semi-group consisting of all these products is free. Then assume that $\{l\}$ may be arranged as an increasing sequence:

$$1 = l_1 < l_2 < l_3 < \ldots < l_n < \ldots .$$

The aim of this paper is to define various functions of the generalised integers having some of the properties of Ramanujan's sum and Nagell's totient function and to develop their properties.

2. Definitions and Ramanujan's sum.

Let l_n, l_r be two generalised integers. Then $l_r \mid l_n$ if there exists l_s, $l_s \in \{l\}$, such that $l_r l_s = l_n$. Let (l_n, l_r) denote the greatest common divisor of l_n and l_r and let $(l_n, l_r)_k$ denote the largest kth power, k an integer, which divides both l_n and l_r.

Now define $c_k(l_n, l_r)$, Ramanujan's sum for generalised integers, by

$$(2.1) \qquad c_k(l_n, l_r) = \sum_{\substack{d \mid l_r \\ d^k \mid l_n}} \mu(l_r/d)\, d^k = \sum_{d^k \mid (l_n, l_r^k)_k} \mu(l_r/d)\, d^k .$$

The function $\mu(l_r)$ is the Möbius function for generalised integers defined by $\mu(l_r) = 0$ if l_r has a square factor; $\mu(l_r) = (-1)^s$ where s denotes the number of prime divisors of l_r and l_r has no square factor; $\mu(1) = 1$. Its properties are given in [6].

The following auxiliary functions will be required. Define

$$(2.2) \qquad \psi_k(l_r) = c_k(0, l_r) = \sum_{d \mid l_r} \mu(l_r/d)\, d^k ,$$

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$$\psi_{k,s}(l_n, l_r) = \sum_{\substack{d \mid l_r \\ (d, l_n) = 1}} \frac{\mu^s(d)}{\psi_k{}^{s-1}(d)}, \tag{2.3}$$

$$\psi_{k,s}(l_r) = l_r{}^s \sum_{d \mid l_r} \frac{\psi_k(d)}{d^s} = \psi_{s,k}(l_r) . \tag{2.4}$$

As the Möbius function is multiplicative, all these functions are multiplicative in l_r.

Let $\gamma(l_n)$ denote the *core* of l_n, that is, if $l_n = p_1{}^{v_1} p_2{}^{v_2} \ldots p_s{}^{v_s}$, then $\gamma(l_n) = p_1 p_2 \ldots p_s$. A generalised integer l_n is called *primitive* if $\gamma(l_n) = l_n$.

Most of the definitions and results in [5] will be needed for this paper. In particular using the corollary to Theorem 6 in [5], it is easily shown that if $d \mid l_r$ and $\delta \mid l_r$ then

$$\psi_k(d)\, c_k(l_n(l_r/d)^k, \delta) = \psi_k(\delta)\, c_k(l_n(l_r/\delta)^k, d) . \tag{2.5}$$

A function $f(l_n, l_r)$ is said to be *k-primitive* if

$$f(l_n, l_r) = f([\gamma(l_n, l_r{}^k)_k]^k, l_r) , \tag{2.6}$$

where the square brackets are used for convenience only. It follows therefore, from the definition of a *k-even* function given in [5, (1.9)] that every k-primitive function is also k-even.

3. The Brauer-Rademacher identity for arithmetical semi-groups.

Recently Eckford Cohen [4], P. J. MacCarthy [7], and M. V. Subbarao [10], have given new proofs of the Brauer-Rademacher identity for natural numbers. In extending the Brauer-Rademacher identity to arithmetical semi-groups the method of Cohen and McCarthy cannot be applied. This is because an essential orthogonality property cannot be extended to the semi-group as addition in it is not defined. However the method of Subbarao is immediately applicable. The basis of his method depends upon finding a multiplicative function $f(r)$ with the additional property that for every prime p,

$$f(p) = f(p^2) = f(p^3) = \ldots .$$

Then taking $f(r) = r/\varphi(r)$ the Brauer-Rademacher identity is obtained. In following through this method for arithmetical semi-groups the Euler φ-function, $\varphi(l_r)$, cannot be used for it can be seen from [6] to be not multiplicative. Instead the function $\psi_k(l_r)$ defined in (2.2) is used with $f(l_r) = l_r{}^k/\psi_k(l_r)$. The method of Subbarao together with result (2.5) then gives the Brauer–Rademacher identity for arithmetical semi-groups as

$$(3.1)\qquad \psi_k(l_r) \sum_{\substack{d|l_r \\ (l_n, d^k)_k=1}} d^k \mu(l_r/d)/\psi_k(d)$$
$$= \mu(l_r) \sum_{d^k|(l_n, l_r^k)_k} d^k \mu(l_r/d) = \mu(l_r)\, c_k(l_n, l_r)\,.$$

When l_r is primitive (3.1) reduces to

$$(3.2)\qquad c_k(l_n, l_r) = \sum_{\substack{d|l_r \\ (l_n, d^k)_k=1}} d^k \mu(d)\, \psi_k(l_r/d)\,.$$

A result more general than the Brauer-Rademacher identity but similar to it is

$$(3.3)\qquad \sum_{\substack{d|l_r \\ (l_n, d^k)_k=1}} \frac{d^k}{\psi_k(d)} h(l_r/d) = \sum_{de=l_r} \frac{g(e)\,\mu(d)}{\psi_k(d)} c_k(l_n, d)\,,$$

where $h(l_r)$ is a given function and $g(l_r) = \sum_{d|l_r} h(d)$.

The proof is obtained by observing that for non-zero terms on the r.h.s. of (3.3), d must be primitive. Substitution of the value for $c_k(l_n, d)$ given by (3.2) in the r.h.s. of (3.3) gives, after some manipulation, the l.h.s. of (3.3).

4. Further properties of Ramanujan's sum.

The first result is similar to Eckford Cohen's Lemma 2.1 in [2]. It should be noted that there is an error in the proof of this lemma, the lemma being untrue if $k \nmid r$. However this does not invalidate the main result of [2].

THEOREM 1. *If l_r, l_i and l_s are generalised integers, l_r is a k-th power but contains no $(k+1)$-th power and $l_i | l_r$ and $l_s | l_r$, then*

$$(4.1)\qquad \sum_{d|l_i} \mu(d) c_k(l_r/d, l_s) = \begin{cases} 0 & \text{if } l_s \text{ contains any } p^3, \\ 0 & \text{if } \gamma(l_i) \nmid \gamma(l_s), \\ P_1{}^k Q^k \mu(Q)\, \psi_k(P_2) & \text{otherwise}\,, \end{cases}$$

where $l_s = P_1 P_2 Q^2$, $P_2 \nmid l_i$ and P_1, P_2, Q are products of single primes. In particular if $k=1$ the sum reduces to $l_i \psi_1(l_s/l_i)$.

PROOF. From (2.1) we have

$$\sum = \sum_{d|l_i} \mu(d)\, c_k(l_r/d, l_s) = \sum_{d|l_i} \mu(d) \sum_{\substack{\delta|l_s \\ \delta^k|l_r/d}} \mu(l_s/\delta)\delta^k = \sum_{\substack{d|l_i,\, \delta|l_s \\ \delta^k|l_r/d}} \delta^k \mu(d)\, \mu(l_s/\delta)\,.$$

Now l_r is a kth power but contains no $(k+1)$th power and so δ must be a product of single primes. Hence if l_s contains any p^3, $\mu(l_s/\delta)=0$ for all δ and the first part of Theorem 1 is proved.

Now put $l_s=PQ^2$ and then l_r is of the form $l_r=P^kQ^kR^k$, $k\geqq 2$. Rewrite $\sum$ as

$$\sum = \sum_{\delta|l_s} \delta^k \mu(l_s/\delta) \sum_{d|(l_i,\, l_r/\delta^k)} \mu(d)\,.$$

The second sum is zero unless $d=1$ [6, Theorem 6]. Hence for a non-zero term $(l_i, l_r/\delta^k)=1$. But $l_i\,|\,l_r$ and so for a non-zero term $\gamma(l_i)\,|\,\delta$ that is $\gamma(l_i)\,|\,\gamma(l_s)$. The sum is therefore zero if $\gamma(l_i)\nmid\gamma(l_s)$ and the second part of the theorem is proved.

Now suppose $\gamma(l_i)\,|\,\gamma(l_s)$ that is $\gamma(l_i)\,|\,PQ$ and so put $l_s=P_1P_2Q_1{}^2Q_2{}^2$ and $l_i=P_1{}^\alpha Q_2{}^\beta$. Then

$$\sum = \sum_{\substack{\delta|l_s\\(l_i,\, l_r/\delta^k)=1}} \delta^k \mu(l_s/\delta) = \sum_{\substack{\delta|PQ\\(P_1{}^\alpha Q_1{}^\beta,\, P^kQ^kR^k/\delta^k)=1}} \delta^k \mu(PQ^2/\delta)\,.$$

Since $\gamma(l_i)\,|\,\delta$ we may write $\delta=P_1Q_1\delta_1$ and then

$$\sum = \sum_{P_1Q_1\delta_1|P_1Q_1P_2Q_2} P_1{}^kQ_1{}^k\delta_1{}^k\mu\left(\frac{P_1P_2Q_1{}^2Q_2{}^2}{P_1Q_1\delta_1}\right)$$

$$= P_1{}^kQ_1{}^k \sum_{\delta_1|P_2Q_2} \delta_1{}^k \mu(P_2Q_2{}^2/\delta_1)\,\mu(Q_1)\,.$$

Again the only non-zero terms are those in which $\delta_1=\delta_2Q_2$. Hence

$$\sum = P_1{}^kQ_1{}^k \sum_{\delta_2|P_2} \delta_2{}^kQ_2{}^k \mu(P_2Q_2/\delta_2)\,\mu(Q_1)$$

$$= P_1{}^kQ_1{}^kQ_2{}^k \mu(Q_1Q_2) \sum_{\delta_2|P_2} \delta_2{}^k \mu(P_2/\delta_2) = P_1{}^kQ^k\mu(Q)\,\psi_k(P_2);$$

this proves the third part of the theorem and when $k=1$,

$$\sum = P_1 \sum_{\delta_2|P_2} \delta_2\, \mu(P_2/\delta_2) = l_i\, \psi_1(l_s/l_i)$$

since $l_s=P_1P_2$, $l_r = P_1P_2R$, $l_i=P_1$. This completes the proof of the theorem.

THEOREM 2.

$$c_k(l_n, l_r) = \sum_{\substack{d|l_r{}^k\\(d,\, l_r{}^k)_k=1}} c(l_n, l_r{}^k/d)\,, \tag{4.2}$$

where $c(l_n,l_r)=c_1(l_n,l_r)$

PROOF. From the definition (2.1) and the method of [10], it can be seen that each side of (4.2) is multiplicative in both l_n and l_r. We need therefore only verify (4.2) for the case $l_n = p^a$, $l_r = p^b$, where p is a generalised prime.

Now

$$c_k(p^a,p^b) = \sum_{\substack{d|p^b \\ d^k|p^a}} \mu(p^b/d)\, d^k = p^{bk} - p^{(b-1)k}, \tag{4.3}$$

where $bk \leqq a$ and $(b-1)k \leqq a$. Also the r.h.s. of (4.2) equals

$$\sum_{\substack{d|p^{bk} \\ (d,p^{bk})_k=1}} \sum_{\substack{\delta|p^{bk}/d \\ \delta|p^a}} \mu\big(p^{bk}/(d\delta)\big)\, \delta .$$

Put $d\delta D = p^{bk}$, then r.h.s. equals

$$\sum_{\substack{d\delta D = p^{bk} \\ \delta|p^a \\ (d,p^{bk})_k=1}} \mu(D)\, \delta .$$

The only non-zero terms occur when $D=1$ or $D=p$. Hence r.h.s. equals

$$\sum_{\substack{d\delta = p^{bk} \\ \delta|p^a \\ (d,p^{bk})_k=1}} \delta - \sum_{\substack{d\delta = p^{bk-1} \\ \delta|p^a \\ (d,p^{bk})_k=1}} \delta = \sum_{\substack{i=bk-k+1 \\ i\leqq a}}^{bk} p^i - \sum_{\substack{i=bk-k \\ i\leqq a}}^{bk-1} p^i = p^{bk} - p^{bk-k},$$

where $bk \leqq a$ and $bk-k \leqq a$. This is just the l.h.s. of (4.2) from (4.3) and so Theorem 2 is proved.

5. k-even functions.

As defined in [5, formula (1.9)] let $f(l_n,l_r)$ and $g(l_n,l_r)$ be k-even functions. Then from [5, Theorems 2 and 4] $f(l_n,l_r)$ may be written uniquely in the form

$$f(l_n,l_r) = \sum_{d|l_r} \alpha(d,l_r)\, c_k(l_n,d) = \sum_{d^k|(l_n,l_r^k)_k} w(d,l_r/d), \tag{5.1}$$

where

$$\begin{aligned} \alpha(d,l_r) &= l_r^{-k} \sum_{\delta|l_r} f\big((l_r/\delta)^k,l_r\big)\, c_k\big((l_r/d)^k,\delta\big) \\ &= l_r^{-k} \sum_{ed|l_r} w(l_r/e,e)\, e^k . \end{aligned} \tag{5.2}$$

(In later work the function $\alpha(d,l_r)$ will sometimes be written $\alpha(d)$). Again $g(l_n,l_r)$ may be written in the form

$$g(l_n,l_r) = \sum_{d|l_r} \beta(d,l_r)\, c_k(l_n,d) .$$

We can now prove

THEOREM 3.

$$\sum_{de=l_r} f(e^k, l_r)\, g(e^k, l_r)\, \psi_k(d) = l_r^k \sum_{d|l_r} \alpha(d, l_r)\, \beta(d, l_r)\, \psi_k(d)\,. \tag{5.3}$$

PROOF. From (5.2) the r.h.s. of (5.3) may be written

$$l_r^k \sum_{d|l_r} l_r^{-k} \sum_{\delta|l_r} f((l_r/\delta)^k, l_r)\, c_k((l_r/d)^k, \delta) \cdot l_r^{-k} \sum_{\Delta|l_r} g((l_r/\Delta)^k, l_r)\, c_k((l_r/d)^k, \Delta)\, \psi_k(d)$$
$$= l_r^{-k} \sum_{\delta|l_r} f((l_r/\delta)^k, l_r) \sum_{\Delta|l_r} g((l_r/\Delta)^k, l_r) \cdot \sum_{d|l_r} c_k((l_r/d)^k, \delta)\, c_k((l_r/d)^k, \Delta)\, \psi_k(d)\,.$$

Now from (2.5) the third sum may be written

$$\sum_{d|l_r} c_k((l_r/d)^k, \delta)\, c_k((l_r/\Delta)^k, d)\, \psi_k(\Delta)$$

and from Theorem 1 of [5] this sum is zero unless $\delta = \Delta$ when its value is $l_r^k \psi_k(\Delta)$. Hence the r.h.s. of (5.3) becomes

$$\sum_{\Delta|l_r} f((l_r/\Delta)^k, l_r)\, g((l_r/\Delta)^k, \delta_r)\, \psi_k(\Delta)$$

and this is just the l.h.s. of (5.3) so Theorem 3 is proved.

THEOREM 4.

$$\sum_{d|l_r} c_k((l_r/d)^k, l_r)\, c_s(l_n, d) = l_r^k (l_n, l_r^s)_s^{(s-k)/s} \psi_{k-s}((l_n, l_r^s)_s^{1/s})\,. \tag{5.4}$$

PROOF. Let $f(l_n, l_r)$ denote the r.h.s. of (5.4). Then

$$l_r^{-s} f(l_n, l_r) = l_r^{k-s} \sum_{d^s|(l_n, l_r^s)_s} \mu(d)/d^{k-s} = \sum_{d^s|(l_n, l_r^s)_s} \mu(d)(l_r/d)^{k-s}\,.$$

The results of (5.1) and (5.2) therefore apply with

$$\alpha(d) = l_r^{-s} \sum_{ed|l_r} \mu(l_r/e)\, e^{k-s}\, e^s = l_r^{-s}\, c_k((l_r/d)^k, l_r)\,.$$

Using (5.1) again, this proves Theorem 4. Putting $l_n = 0$ in Theorem 4 we obtain

$$\sum_{d|l_r} c_k((l_r/d)^k, l_r)\, \psi_s(d) = l_r^s\, \psi_{k-s}(l_r)\,. \tag{5.5}$$

6. k-primitive functions.

THEOREM 5. *A function $f(l_n, l_r)$ is k-primitive if and only if it can be written in the form*

$$f(l_n, l_r) = \sum_{d|\gamma(l_r)} \alpha(d, l_r)\, c_k(l_n, d)\,. \tag{6.1}$$

PROOF. Suppose first that $f(l_n, l_r)$ is k-primitive and so k-even and therefore the results of (5.1) and (5.2) apply. In this case

$$d^k|(l_n, l_r^k)_k \Leftrightarrow d^k|([\gamma(l_n, l_r^k)_k]^k, l_r^k)_k \Leftrightarrow d^k|\,[\gamma(l_n, l_r^k)_k]^k .$$

Hence

$$f(l_n, l_r) = \sum_{d^k|(l_n, l_r^k)_k} |\mu(d)|\; w(d, l_r/d)$$

and

$$\alpha(d) = l_r^{-k} \sum_{e|l_r/d} w(l_r/e, e)e^k\,|\mu(l_r/e)| .$$

But if d contains a square factor D^2, then D^2 is also a factor of l_r/e for all divisors e of l_r/d. Therefore $\alpha(d)=0$ unless d is primitive and so the necessity half of Theorem 5 is proved. Now suppose $f(l_n, l_r)$ may be written in the form (6.1). Then

$$\begin{aligned} f([\gamma(l_n, l_r^k)_k]^k, l_r) &= \sum_{d|\gamma(l_r)} \alpha(d)\; c_k([\gamma(l_n, l_r^k)_k]^k, d) \\ &= \sum_{d|\gamma(l_r)} \alpha(d)\; c_k((l_n, l_r^k)_k, d) \\ &= \sum_{d|\gamma(l_r)} \alpha(d)\; c_k(l_n, d) . \end{aligned}$$

This completes the proof of Theorem 5.

THEOREM 6. *$f(l_n, l_r)$ is k-primitive if and only if it can be written in the form*

$$f(l_n, l_r) = \sum_{\substack{d|\gamma(l_r) \\ (l_n, d^k)_k=1}} h(d, \gamma(l_r)/d) . \tag{6.2}$$

PROOF. When d is primitive,

$$(l_n, d^k)_k = 1 \Leftrightarrow ([\gamma(l_n, l_r^k)_k]^k, d^k)_k = 1 ,$$

and so if $f(l_n, l_r)$ can be written in the form (6.2),

$$f(l_n, l_r) = f([\gamma\,(l_n, l_r^k)_k]^k, l_r) ,$$

and the function is k-primitive. Now suppose $f(l_n, l_r)$ is k-primitive and therefore may be written in the form (6.1). Then use of the result (3.2) in the expression for $c_k(l_n, d)$ eventually yields (6.2) with

$$h(d, \gamma(l_r)/d) = d^k \mu(d) \sum_{\delta|\gamma(l_r)/d} \alpha(\delta d)\; \psi_k(\delta) .$$

This completes the proof of Theorem 6.

7. Nagell's totient function.

Eckford Cohen in [3] has defined the function $\varphi^{(s)}(n,r)$, where s, n and r are positive integers, as follows: $\varphi^{(s)}(n,r)$ is the number of solutions $x_i \pmod r$, $(x_i,r)=1$ of the congruence $n \equiv x_1+x_2+\ldots+x_s \pmod r$. When $s=2$, the function becomes Nagell's totient function. In [3], Cohen has also shown that

$$\varphi^{(s)}(n,r) = r^{-1} \sum_{d\delta=r} c^s(\delta,r)\, c(n,d),$$

where $c(n,r)$ is Ramanujan's sum. We therefore now define

$$\theta_{k,s}(l_n,l_r) = l_r^{-k} \sum_{d|l_r} c_k{}^s((l_r/d)^k,l_r)\, c_k(l_n,d)\,. \tag{7.1}$$

Applying [5, Theorem 6, Corollary] we have

$$c_k((l_r/d)^k,l_r) = \frac{\psi_k(l_r)\,\mu(d)}{\psi_k(d)}.$$

Hence

$$\theta_{k,s}(l_n,l_r) = \frac{\psi_k{}^s(l_r)}{l_r^k} \sum_{d|l_r} \frac{\mu^s(d)}{\psi_k{}^s(d)} c_k(l_n,d)\,. \tag{7.2}$$

Because of the Möbius function this may be written

$$\theta_{k,s}(l_n,l_r) = \frac{\psi_k{}^s(l_r)}{l_r^k} \sum_{d|\gamma(l_r)} \frac{\mu^s(d)}{\psi_k{}^s(d)} c_k(l_n,d)\,. \tag{7.3}$$

From Theorem 5 we can see therefore that $l_r^k\,\theta_{k,s}(l_n,l_r)/\psi_k{}^s(l_r)$ is a k-primitive function. Here (7.3) gives us the value for $\alpha(d)$ and applying Theorem 6, with the value for $h(d,\gamma(l_r)/d)$ given by (6.3) gives

$$\frac{l_r^k}{\psi_k{}^s(l_r)}\,\theta_{k,s}(l_n,l_r) = \sum_{\substack{d|\gamma(l_r)\\ (l_n,d^k)_k=1}} \frac{d^k\mu^{s+1}(d)}{\psi_k{}^s(d)} \sum_{\delta|(\gamma(l_r)/d)} \frac{\mu^s(\delta)}{\psi_k{}^{s-1}(\delta)}.$$

But from the definition (2.3), the second sum equals $\psi_{k,s}(1,\gamma(l_r)/d)$ and so

$$\theta_{k,s}(l_n,l_r) = \frac{\psi_k{}^s(l_r)}{l_r^k} \sum_{\substack{d|\gamma(l_r)\\ (l_n,d^k)_k=1}} \frac{d^k\mu^{s+1}(d)}{\psi_k{}^s(d)}\,\psi_{k,s}\left(1,\frac{\gamma(l_r)}{d}\right). \tag{7.4}$$

This general formula only becomes analogous to Cohen's evaluation of Nagell's totient function in the special case $s=2$.

THEOREM 7.

$$\theta_{k,s}(l_n,l_r) = l_r^{-k} \sum_{d^k|(l_n,l_r^k)_k} d^k\, c_k{}^s((l_r/d)^k,l_r)\,\psi_{k,s+1}(d,l_r)\,.$$

PROOF. Put

$$F(l_n, l_r) = \frac{l_r^k}{\psi_k^s(l_r)}\,\theta_{k,s}(l_n, l_r) = \sum_{d|l_r} \frac{\mu^s(d)}{\psi_k^s(d)}\,c_k(l_n, d)\,, \tag{7.5}$$

the last equality being due to (7.2). From (5.1) we conclude that $F(l_n, l_r)$ is k-even with its $\alpha(d)$ given by (7.5). Hence inversion of (5.2) in the usual way gives the corresponding value for $w(d, l_r/d)$. Substitution of this value for $w(d, l_r/d)$ back in (5.1) gives

$$\begin{aligned}\theta_{k,s}(l_n, l_r) &= \frac{\psi_k^s(l_r)}{l_r^k} \sum_{d^k|(l_n, l_r^k)_k} d^k \sum_{e|l_r/d} \frac{\mu(l_r/(de))\,\mu^s(l_r/e)}{\psi_k^s(l_r/e)} \\ &= \frac{\psi_k^s(l_r)}{l_r^k} \sum_{\substack{d^k|l_n \\ de\Delta = l_r}} \frac{d^k\,\mu(\Delta)\,\mu^s(d\Delta)}{\psi_k^s(d\Delta)}\,.\end{aligned}$$

For non-zero terms Δ must be primitive and $d\Delta$ primitive and so $(d, \Delta) = 1$ and d is also primitive. Therefore

$$\begin{aligned}\theta_{k,s}(l_n, l_r) &= l_r^{-k} \sum_{\substack{d^k|l_n \\ d|l_r}} \frac{d^k\,\psi_k^s(l_r)\,\mu^s(d)}{\psi_k^s(d)} \sum_{\Delta|l_r/d} \frac{\mu^{s+1}(\Delta)}{\psi_k^s(\Delta)} \\ &= l_r^{-k} \sum_{d^k|(l_n, l_r^k)_k} d^k\,c_k^s((l_r/d)^k, l_r)\,\psi_{k,s+1}(1, l_r/d)\end{aligned}$$

from [5, Theorem 6, Corollary]. Now

$$\psi_{k,s+1}(d, l_r) = \sum_{\substack{\delta|l_r \\ (\delta,d)=1}} \frac{\mu^{s+1}(\delta)}{\psi_k^s(\delta)}$$

and $d\,|\,l_r$ also, and so $(\delta, d) = 1$ only if $\delta\,|\,l_r/d$, and in this particular case

$$\psi_{k,\,s+1}(d, l_r) = \psi_{k,\,s+1}(1, l_r/d)\,.$$

This completes the proof.

Again, applying [5, Theorem 7 (the inversion formula)] to (7.1) gives

$$c_k^s(l_n, l_r) = \sum_{d|l_r} \theta_{k,\,s}((l_r/d)^k, l_r)\,c_k(l_n, d)\,. \tag{7.6}$$

Putting $l_n = 0$ in (7.6) we obtain

$$\psi_k^s(l_r) = \sum_{d|l_r} \theta_{k,\,s}((l_r/d)^k, l_r)\,\psi_k(d)\,. \tag{7.7}$$

Putting $l_n = 1$ in (7.6) we have $c_k(1, l_r) = \mu(l_r)$ and

$$\mu^s(l_r) = \sum_{d|l_r} \theta_{k,\,s}((l_r/d)^k, l_r)\,\mu(d)\,. \tag{7.8}$$

Another property of $\theta_{k,s}(l_n, l_r)$ may be obtained by using Theorem 3. Put $f(l_n, l_r) = c_k(l_n, l_r)$ and $g(l_n, l_r) = \theta_{k,s}(l_n, l_r)$. Then $c_k(l_n, l_r) = \sum_{d|l_r} \alpha(d, l_r)\, c_k(l_n, d)$.

Hence in this case

$$\alpha(d, l_r) = \begin{cases} 0 & \text{if } d < l_r \\ 1 & \text{if } d = l_r \end{cases}.$$

Also from (7.2),

$$\beta(d, l_r) = \frac{\psi_k{}^s(l_r)}{l_r{}^k} \frac{\mu^s(d)}{\psi_k{}^s(d)}.$$

From Theorem 3 we therefore have

$$\sum_{de=l_r} c_k(e^k, l_r)\, \theta_{k,s}(e^k, l_r)\, \psi_k(d) = l_r{}^k \frac{\psi_k{}^s(l_r)}{l_r{}^k} \frac{\mu^s(l_r)}{\psi_k{}^s(l_r)} \psi_k(l_r) = \mu^s(l_r)\, \psi_k(l_r)\,. \tag{7.9}$$

Special case $s = 2$. Putting $s = 2$ in (7.4) we have

$$\theta_{k,2}(l_n, l_r) = \frac{\psi_k{}^2(l_r)}{l_r{}^k} \sum_{\substack{d|\gamma(l_r) \\ (l_n, d^k)_k = 1}} \frac{d^k \mu^3(d)}{\psi_k{}^2(d)} \psi_{k,2}\left(1, \frac{\gamma(l_r)}{d}\right).$$

Now $\psi_{k,2}(1, l_r)$ is multiplicative in l_r and we need therefore only evaluate it for the prime power p^a:

$$\psi_{k,2}(1, p^a) = \sum_{d|p^a} \frac{\mu^2(d)}{\psi_k(d)} = \frac{1}{\psi_k(1)} + \frac{1}{\psi_k(p)} = 1 + \frac{1}{p^k - 1} = \frac{p^k}{\psi_k(p)}.$$

Hence

$$\psi_{k,2}\left(1, \frac{\gamma(l_r)}{d}\right) = \frac{\gamma^k(l_r)}{d^k} \frac{\gamma_k(d)}{\psi_k(\gamma(l_r))}.$$

However it is easily verified from (2.2) that

$$\frac{\psi_k(l_r)}{l_r{}^k} = \frac{\psi_k(\gamma(l_r))}{\gamma^k(l_r)}, \tag{7.10}$$

and so

$$\theta_{k,2}(l_n, l_r) = \psi_k(l_r) \sum_{\substack{d|\gamma(l_r) \\ (l_n, d^k)_k = 1}} \frac{\mu(d)}{\psi_k(d)}.$$

Because of the Möbius function we therefore have

$$\theta_{k,2}(l_n, l_r) = \psi_k(l_r) \sum_{\substack{d|l_r \\ (l_n, d^k)_k = 1}} \frac{\mu(d)}{\psi_k(d)}. \tag{7.11}$$

This formula is analogous to Cohen's evaluation of Nagell's totient function.

8. Another analogue of Nagell's totient function.

P. J. McCarthy in [8] has defined the function $N_k^*(n,r,t)$ to be the number of solutions $(\bmod r^k)$ of the congruence

$$n \equiv x_1+x_2+\ldots+x_t \pmod{r^k},$$

where $(x_1,x_2,\ldots x_t,r^k)_k=1$, and quoted the result

$$N_k^*(n,r,t) = r^{-k}\sum_{d\mid r} c_{kt}((r/d)^{kt},r)\, c_k(n,d)\,.$$

When $t=2$ we have another analogue of Nagell's totient function. We therefore define

$$\theta^*_{k,s}(l_n,l_r) = l_r^{-k}\sum_{d\mid l_r} c_{ks}((l_r/d)^{ks},l_r)\, c_k(l_n,d)\,. \tag{8.1}$$

Applying Theorem 4 to this definition we therefore have

$$\theta^*_{k,s}(l_n,l_r) = l_r^{k(s-1)}(l_n,l_r^k)_k^{1-s}\,\psi_{ks-k}((l_n,l_r^k)_k^{1/k})\,. \tag{8.2}$$

Applying [5, Theorem 6, Corollary], we have

$$c_{ks}((l_r/d)^{ks},l_r) = \frac{\psi_{ks}(l_r)\,\mu(d)}{\psi_{ks}(d)}$$

and so from (8.1)

$$\begin{aligned}\theta^*_{k,s}(l_n,l_r) &= \frac{\psi_{ks}(l_r)}{l_r^k}\sum_{d\mid l_r}\frac{\mu(d)\,c_k(l_n,d)}{\psi_{ks}(d)}\\ &= \frac{\psi_{ks}(l_r)}{l_r^k}\sum_{d\mid\gamma(l_r)}\frac{\mu(d)\,c_k(l_n,d)}{\psi_{ks}(d)}.\end{aligned} \tag{8.3}$$

From Theorem 5 it follows that

$$\frac{l_r^k\,\theta^*_{k,s}(l_n,l_r)}{\psi_{ks}(l_r)}$$

is k-primitive with $\alpha(d)=\mu(d)/\psi_{ks}(d)$. From Theorem 6 we may therefore write

$$\begin{aligned}\theta^*_{k,s}(l_n,l_r) &= \frac{\psi_{ks}(l_r)}{l_r^k}\sum_{\substack{d\mid\gamma(l_r)\\ (l_n,d^k)_k=1}} d^k\,\mu(d)\sum_{\delta\mid(\gamma(l_r)/d)}\frac{\mu(\delta d)}{\psi_{ks}(\delta d)}\psi_k(\delta)\\ &= \frac{\psi_{ks}(l_r)}{l_r^k}\sum_{\substack{d\mid\gamma(l_r)\\ (l_n,d^k)_k=1}}\frac{d^k\,\mu^2(d)}{\psi_{ks}(d)}\sum_{\delta\mid(\gamma(l_r)/d)}\frac{\mu(\delta)\,\psi_k(\delta)}{\psi_{ks}(\delta)}.\end{aligned}$$

However from Example 5 in [9] and [5] we have

$$\sum_{d|l_r} \frac{\mu(d)\,\psi_k(d)}{\psi_t(d)} = \frac{l_r{}^k\ \psi_{t-k}(l_r)}{\psi_t(l_r)}, \qquad t-k \neq 0\,. \tag{8.4}$$

Thus

$$\sum_{\delta|(\gamma(l_r)/d)} \frac{\mu(\delta)\,\psi_k(\delta)}{\gamma_{ks}(\delta)} = \frac{\gamma^k(l_r)}{d^k}\,\frac{\psi_{ks-k}(\gamma(l_r)/d)}{\psi_{ks}(\gamma(l_r)/d)}.$$

Hence

$$\begin{aligned} \theta^*_{k,s}(l_n,l_r) &= \frac{\psi_{ks}(l_r)}{l_r{}^k}\gamma^k(l_r)\,\frac{\psi_{ks-k}(\gamma(l_r))}{\psi_{ks}(\gamma(l_r))} \sum_{\substack{d|\gamma(l_r)\\ (l_n,d^k)_k=1}} \frac{\mu^2(d)}{\psi_{ks-k}(d)} \\ &= \psi_{ks-k}(l_r) \sum_{\substack{d|\gamma(l_r)\\ (l_n,d^k)_k=1}} \frac{\mu^2(d)}{\psi_{ks-k}(d)} \end{aligned}$$

from (7.10). We may therefore write

$$\theta^*_{k,s}(l_n,l_r) = \psi_{ks-k}(l_r) \sum_{\substack{d|l_r\\ (l_n,d^k)_k=1}} \frac{\mu^2(d)}{\psi_{ks-k}(d)}. \tag{8.5}$$

This general formula is analogous to McCarthy's evaluation of Nagell's totient function.

From definition (8.1) it can be seen that $\theta^*_{k,s}(l_n,l_r)$ is k-even and so the inversion formula, Theorem 7 of [5], may be applied. In (8.1) put $ks=t$ and

$$f(l_n,l_r) = l_r{}^k\,\theta^*_{k,s}(l_n,l_r) = \sum_{d|l_r} c_t((l_r/d)^t,l_r)\,c_k(l_n,d)$$

and then $w(d,l_r)=c_t((l_r/d)^t,l_r)$. Hence

$$w(l_m,l_r) = c_t((l_r/l_m)^t,l_r) \quad \text{and} \quad l_m{}^k = \frac{l_r{}^k}{(l_n,l_r{}^k)_k}.$$

Thus

$$(l_r/l_m)^t = \left((l_r/l_r)(l_n,l_r{}^k)_k^{1/k}\right)^t = (l_n,l_r{}^k)_k^{t/k}\,.$$

Hence

$$w(l_m,l_r) = c_t((l_n,l_r{}^k)_k^{t/k},l_r) = c_{ks}((l_n,l_r{}^k)_k{}^s,l_r) = c_{ks}(l_n{}^s,l_r)$$

by the definition of c_{ks}. Hence

$$\begin{aligned} c_{ks}(l_n{}^s,l_r) &= l_r{}^{-k}\sum_{d|l_r} l_r{}^k\theta^*_{k,s}((l_r/d)^k,l_r)\,c_k(l_n,d) \\ &= \sum_{d|l_r} \theta^*_{k,s}((l_r/d)^k,l_r)\,c_k(l_n,d)\,. \end{aligned} \tag{8.6}$$

In the special case $l_n = 0$ we have

$$(8.7) \qquad \psi_{ks}(l_r) = \sum_{d|l_r} \theta^*_{k,s}((l_r/d)^k, l_r)\, \psi_k(d) .$$

REFERENCES

1. A. Beurling, *Analyse de la loi asymptotique de la distribution des nombres premiers généralisés*, Acta Math. 68 (1937), 255–291.
2. Eckford Cohen, *Nagell's totient function*, Math. Scand. 8 (1960), 55–58.
3. Eckford Cohen, *Representations of even functions* (mod r). I, *Arithmetical identities*, Duke Math. J. 25 (1958), 401–421.
4. Eckford Cohen, *Representations of even functions* (mod r). II, *Cauchy products*, Duke Math. J. 26 (1959), 165–182.
5. E. M. Horadam, *Ramanujan's sum for generalised integers*, Duke Math. J. 31 (1964), 697–702.
6. E. M. Horadam, *Arithmetical functions of generalized primes*, Amer. Math. Month. 68 (1961), 626–629.
7. P. J. McCarthy, *Some remarks on arithmetical identities*, Amer. Math. Month. 67 (1960), 539–548.
8. P. J. MacCarthy, *Some more remarks on arithmetical identities*, Portugaliae Math. 21 (1962), 45–57.
9. P. J. MacCarthy, *The generation of arithmetical identities*, J. Reine Angew. Math. 203 (1960), 55–63.
10. M. V. Subbarao, *The Brauer-Rademacher identity*, Amer. Math. Month. 72 (1965), 135–138.

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