RAMANUJAN’S SUM AND NAGELL’S TOTIENT FUNCTION FOR ARITHMETICAL SEMI-GROUPS

E. M. HORADAM

1. Introduction.

The arithmetical semi-group for this work was first defined by Arne Beurling in [1]; here it is defined as follows. Suppose there is given a finite or infinite sequence \( \{p\} \) of real numbers, called generalised primes, such that \( 0 < 1 < p_1 < p_2 < \ldots \). Form the set \( \{1\} \) of all \( p \)-products, that is, products \( p_1 v_1 p_2 v_2 \ldots \), where \( v_1, v_2, \ldots \), are integers \( \geq 0 \) of which all but a finite number are 0. Call these numbers generalised integers and suppose that the semi-group consisting of all these products is free. Then assume that \( \{1\} \) may be arranged as an increasing sequence:

\[
1 = l_1 < l_2 < l_3 < \ldots < l_n < \ldots
\]

The aim of this paper is to define various functions of the generalised integers having some of the properties of Ramanujan’s sum and Nagell’s totient function and to develop their properties.

2. Definitions and Ramanujan’s sum.

Let \( l_n, l_r \) be two generalised integers. Then \( l_r | l_n \) if there exists \( l_s, l_s \in \{1\}, \) such that \( l_s l_r = l_n \). Let \( (l_n, l_r) \) denote the greatest common divisor of \( l_n \) and \( l_r \) and let \( (l_n, l_r)_k \) denote the largest \( k \)th power, \( k \) an integer, which divides both \( l_n \) and \( l_r \).

Now define \( c_k(l_n, l_r) \), Ramanujan’s sum for generalised integers, by

\[
c_k(l_n, l_r) = \sum_{d \mid (l_n, l_r)_k} \mu(l_r/d) d^k = \sum_{d \mid (l_n, l_r)_k} \mu(l_r/d) d^k.
\]

The function \( \mu(l_r) \) is the Möbius function for generalised integers defined by \( \mu(l_r) = 0 \) if \( l_r \) has a square factor; \( \mu(l_r) = (-1)^s \) where \( s \) denotes the number of prime divisors of \( l_r \) and \( l_r \) has no square factor; \( \mu(1) = 1 \). Its properties are given in [6].

The following auxiliary functions will be required. Define

\[
\psi_k(l_r) = c_k(0, l_r) = \sum_{d \mid l_r} \mu(l_r/d) d^k,
\]

Received April 7, 1967; revised December 6, 1967.
\(2.3\)

\[ \psi_{k,s}(l_n, l_r) = \sum_{d \mid l_r, \mu_s(d)} \frac{\mu_s(d)}{\psi_{k,s-1}(d)}, \]

\(2.4\)

\[ \psi_{k,s}(l_r) = l_r^s \sum_{d \mid l_r} \frac{\psi_k(d)}{d^s} = \psi_{s,k}(l_r). \]

As the Möbius function is multiplicative, all these functions are multiplicative in \(l_r\).

Let \(\gamma(l_n)\) denote the core of \(l_n\), that is, if \(l_n = p_1^{v_1} p_2^{v_2} \ldots p_s^{v_s}\), then \(\gamma(l_n) = p_1 p_2 \ldots p_s\). A generalised integer \(l_n\) is called primitive if \(\gamma(l_n) = l_n\).

Most of the definitions and results in [5] will be needed for this paper. In particular using the corollary to Theorem 6 in [5], it is easily shown that if \(d \mid l_r\) and \(\delta \mid l_r\) then

\(2.5\)

\[ \psi_k(d) c_k(l_n(l_r/d)^k, \delta) = \psi_k(\delta) c_k(l_n(l_r/\delta)^k, d). \]

A function \(f(l_n, l_r)\) is said to be \(k\)-primitive if

\(2.6\)

\[ f(l_n, l_r) = f([\gamma(l_n, l_r)^k, l_r], \delta). \]

where the square brackets are used for convenience only. It follows therefore, from the definition of a \(k\)-even function given in [5, (1.9)] that every \(k\)-primitive function is also \(k\)-even.

3. The Brauer-Rademacher identity for arithmetical semi-groups.

Recently Eckford Cohen [4], P. J. MacCarthy [7], and M. V. Subbarao [10], have given new proofs of the Brauer-Rademacher identity for natural numbers. In extending the Brauer-Rademacher identity to arithmetical semi-groups the method of Cohen and MacCarthy cannot be applied. This is because an essential orthogonality property cannot be extended to the semi-group as addition in it is not defined. However, the method of Subbarao is immediately applicable. The basis of his method depends upon finding a multiplicative function \(f(r)\) with the additional property that for every prime \(p\),

\[ f(p) = f(p^2) = f(p^3) = \ldots. \]

Then taking \(f(r) = r/\varphi(r)\) the Brauer-Rademacher identity is obtained. In following through this method for arithmetical semi-groups the Euler \(\varphi\)-function, \(\varphi(l_r)\), cannot be used for it can be seen from [6] to be not multiplicative. Instead the function \(\psi_k(l_r)\) defined in (2.2) is used with \(f(l_r) = l_r^k/\psi_k(l_r)\). The method of Subbarao together with result (2.5) then gives the Brauer–Rademacher identity for arithmetical semi-groups as
\[ (3.1) \quad \sum_{\substack{d \mid l_r \atop (d_n, d^2) = 1}} \frac{d^k \mu(l_r/d)}{\varphi_k(d)} = \mu(l_r) \sum_{d^k | (l_n, l_r)} d^k \mu(l_r/d) = \mu(l_r) c_k(l_n, l_r). \]

When \( l_r \) is primitive (3.1) reduces to
\[ (3.2) \quad c_k(l_n, l_r) = \sum_{\substack{d \mid l_r \atop (d_n, d) = 1}} d^k \mu(d) \varphi_k(l_r/d). \]

A result more general than the Brauer-Rademacher identity but similar to it is
\[ (3.3) \quad \sum_{\substack{d \mid l_r \atop (d_n, d^2) = 1}} \frac{d^k}{\varphi_k(d)} h(l_r/d) = \sum_{\substack{d \mid l_r \atop (d_n, d) = 1}} \frac{g(c) \mu(d)}{\varphi_k(d)} c_k(l_n, d), \]

where \( h(l_r) \) is a given function and \( g(l_r) = \sum_{d \mid l_r} h(d) \).

The proof is obtained by observing that for non-zero terms on the r.h.s. of (3.3), \( d \) must be primitive. Substitution of the value for \( c_k(l_n, d) \) given by (3.2) in the r.h.s. of (3.3) gives, after some manipulation, the l.h.s. of (3.3).

4. Further properties of Ramanujan's sum.

The first result is similar to Eckford Cohen's Lemma 2.1 in [2]. It should be noted that there is an error in the proof of this lemma, the lemma being untrue if \( k \nmid r \). However this does not invalidate the main result of [2].

**Theorem 1.** If \( l_r, l_i \) and \( l_s \) are generalised integers, \( l_r \) is a \( k \)-th power but contains no \( (k + 1) \)-th power and \( l_i \mid l_r \) and \( l_s \mid l_r \), then

\[ (4.1) \quad \sum_{d \mid l_i} \mu(d) c_k(l_r/d, l_s) = \begin{cases} 0 & \text{if } l_s \text{ contains any } p^3, \\ 0 & \text{if } \gamma(l_i) \nmid \gamma(l_s), \\ P_1^k Q^k \mu(Q) \varphi_k(P_2) & \text{otherwise}, \end{cases} \]

where \( l_s = P_1 P_2 Q^2 \), \( P_2 \nmid l_i \) and \( P_1, P_2, Q \) are products of single primes. In particular if \( k = 1 \) the sum reduces to \( l_i \varphi_1(l_s/l_i) \).

**Proof.** From (2.1) we have
\[ \sum_{d \mid l_i} \mu(d) c_k(l_r/d, l_s) = \sum_{d \mid l_i} \mu(d) \sum_{\substack{\delta \mid l_s \atop (\delta, l_r) = 1}} \mu(l_s/\delta) \delta^k = \sum_{\substack{\delta \mid l_s \atop (\delta, l_r) = 1}} \delta^k \mu(\delta) \mu(l_s/\delta). \]

Now \( l_r \) is a \( k \)th power but contains no \( (k + 1) \)th power and so \( \delta \) must be
a product of single primes. Hence if \( l_s \) contains any \( p^a, \mu(l_s/\delta) = 0 \) for all \( \delta \) and the first part of Theorem 1 is proved.

Now put \( l_s = PQ^2 \) and then \( l_r \) is of the form \( l_r = P^kQ^kR^k, k \geq 2 \). Rewrite \( \Sigma \) as

\[
\Sigma = \sum_{\delta \mid l_s} \delta^k \mu(l_s/\delta) \sum_{d \mid (l_i, l_r/\delta^k)} \mu(d) .
\]

The second sum is zero unless \( d = 1 \) [6, Theorem 6]. Hence for a non-zero term \((l_i, l_r/\delta^k) = 1\). But \( l_i \mid l_r \) and so for a non-zero term \( \gamma(l_i) \mid \delta \) that is \( \gamma(l_i) \mid \gamma(l_s) \). The sum is therefore zero if \( \gamma(l_i) \not\mid \gamma(l_s) \) and the second part of the theorem is proved.

Now suppose \( \gamma(l_i) \mid \gamma(l_s) \) that is \( \gamma(l_i) \mid PQ \) and so put \( l_s = P_1P_2Q_1^2Q_2^2 \) and \( l_i = P_1^\alpha Q_2^\beta \). Then

\[
\sum = \sum_{\delta \mid l_s} \delta^k \mu(l_s/\delta) = \sum_{\delta \mid PQ} \delta^k \mu(PQ^2/\delta) .
\]

Since \( \gamma(l_i) \mid \delta \) we may write \( \delta = P_1Q_1^2 = 1 \) and then

\[
\sum = \sum_{P_1Q_1P_2P_3} P_1^kQ_1^kP_2P_3 \delta_1^k \mu \left( \frac{P_1P_2Q_1^2Q_2^2}{P_1Q_1^2} \right) = P_1^kQ_1^k \sum_{\delta_1 \mid P_2P_3} \delta_1^k \mu(P_2Q_2^2/\delta_1) \mu(Q_1) .
\]

Again the only non-zero terms are those in which \( \delta_1 = \delta_2Q_2 \). Hence

\[
\sum = P_1^kQ_1^k \sum_{\delta_2 \mid P_2} \delta_2^kQ_2^k \mu(P_2Q_2/\delta_2) \mu(Q_1)
\]

\[
= P_1^kQ_1^kQ_2^k \mu(Q_1Q_2) \sum_{\delta_2 \mid P_2} \delta_2^k \mu(P_2/\delta_2) = P_1^kQ_1^k \mu(Q) \psi_2(P_2);
\]

this proves the third part of the theorem and when \( k = 1 \),

\[
\sum = P_1 \sum_{\delta_2 \mid P_2} \delta_2 \mu(P_2/\delta_2) = \psi_2(l_r/l_i)
\]

since \( l_s = P_1P_2, l_r = P_1P_2R, l_i = P_1 \). This completes the proof of the theorem.

**Theorem 2.**

\[(4.2)\]

\[
c_k(l_n, l_r) = \sum_{d \mid l_n^k} c(l_n, l_r/k/d),
\]

where \( c(l_n, l_r) = c_1(l_n, l_r) \)

**Proof.** From the definition (2.1) and the method of [10], it can be seen that each side of (4.2) is multiplicative in both \( l_n \) and \( l_r \). We need
therefore only verify (4.2) for the case $l_n = p^a$, $l_r = p^b$, where $p$ is a generalised prime.

Now

\[(4.3) \quad c_k(p^a, p^b) = \sum_{d \mid p^b} \mu(p^b/d) d^k = p^{bk} - p^{(b-1)k},\]

where $bk \leq a$ and $(b-1)k \leq a$. Also the r.h.s. of (4.2) equals

\[\sum_{d \mid p^b k \delta \mid p^a} \sum_{\delta \mid p^a} \mu(p^{bk}/(d\delta)) \delta.\]

Put $d\delta D = p^{bk}$, then r.h.s. equals

\[\sum_{d\delta D = p^{bk}} \mu(D) \delta.\]

The only non-zero terms occur when $D = 1$ or $D = p$. Hence r.h.s. equals

\[\sum_{d\delta = p^{bk}} \delta - \sum_{d\delta = p^{bk-1}} \delta = \sum_{i=bk-k+1}^{bk} p^i - \sum_{i=bk-k}^{bk-1} p^i = p^{bk} - p^{bk-k},\]

where $bk \leq a$ and $bk - k \leq a$. This is just the l.h.s. of (4.2) from (4.3) and so Theorem 2 is proved.

5. $k$-even functions.

As defined in [5, formula (1.9)] let $f(l_n, l_r)$ and $g(l_n, l_r)$ be $k$-even functions. Then from [5, Theorems 2 and 4] $f(l_n, l_r)$ may be written uniquely in the form

\[(5.1) \quad f(l_n, l_r) = \sum_{d \mid l_r} \alpha(d, l_r) c_k(l_n, d) = \sum_{d \mid [l_n, l_r]_k} w(d, l_r/d),\]

where

\[(5.2) \quad \alpha(d, l_r) = l_r^{-k} \sum_{\delta \mid l_r} f((l_r/\delta)^k, l_r) c_k((l_r/\delta)^k, \delta) = l_r^{-k} \sum_{\delta \mid l_r} w(l_r/\delta, e) e^k.\]

(In later work the function $\alpha(d, l_r)$ will sometimes be written $\alpha(d)$). Again $g(l_n, l_r)$ may be written in the form

\[g(l_n, l_r) = \sum_{d \mid l_r} \beta(d, l_r) c_k(l_n, d).\]
We can now prove

**Theorem 3.**

\[(5.3) \quad \sum_{d \in \mathbb{I}_r} f(e^k, l_r) g(e^k, l_r) \psi_k(d) = l_r^k \sum_{d \mid l_r} \alpha(d, l_r) \beta(d, l_r) \psi_k(d).\]

**Proof.** From (5.2) the r.h.s. of (5.3) may be written

\[l_r^k \sum_{d \mid l_r} l_r^{-k} \sum_{\delta \mid l_r} f((l_r/\delta)^k, l_r) c_k((l_r/d)^k, \delta) \cdot l_r^{-k} \sum_{\Delta \mid l_r} g((l_r/\Delta)^k, l_r) c_k((l_r/d)^k, \Delta) \psi_k(d)\]

\[= l_r^{-k} \sum_{\delta \mid l_r} f((l_r/\delta)^k, l_r) \sum_{\Delta \mid l_r} g((l_r/\Delta)^k, l_r) \cdot \sum_{d \mid l_r} c_k((l_r/d)^k, \delta) c_k((l_r/d)^k, \Delta) \psi_k(d).\]

Now from (2.5) the third sum may be written

\[\sum_{d \mid l_r} c_k((l_r/d)^k, \delta) c_k((l_r/\Delta)^k, d) \psi_k(\Delta)\]

and from Theorem 1 of [5] this sum is zero unless \(\delta = \Delta\) when its value is \(l_r^k \psi_k(\Lambda)\). Hence the r.h.s. of (5.3) becomes

\[\sum_{d \mid l_r} f((l_r/\Delta)^k, l_r) g((l_r/\Delta)^k, \delta_r) \psi_k(\Delta)\]

and this is just the l.h.s. of (5.3) so Theorem 3 is proved.

**Theorem 4.**

\[(5.4) \quad \sum_{d \mid l_r} c_k((l_r/d)^k, l_r) c_s(l_n, d) = l_r^k (l_n, l_r^s)^{(s-k)/s} \psi_{k-s}(l_n, l_r^s)^{1/s}).\]

**Proof.** Let \(f(l_n, l_r)\) denote the r.h.s. of (5.4). Then

\[l_r^{-s} f(l_n, l_r) = l_r^{-k-s} \sum_{d \mid (l_n, l_r)^s} \mu(d)/d^{k-s} = \sum_{d \mid (l_n, l_r)^s} \mu(d)(l_r/d)^{k-s}.\]

The results of (5.1) and (5.2) therefore apply with

\[\alpha(d) = l_r^{-s} \sum_{e \mid d} \mu(l_r/e) e^{k-s} e^s = l_r^{-s} c_k((l_r/d)^k, l_r).\]

Using (5.1) again, this proves Theorem 4. Putting \(l_n = 0\) in Theorem 4 we obtain

\[(5.5) \quad \sum_{d \mid l_r} c_k((l_r/d)^k, l_r) \psi_s(d) = l_r^s \psi_{k-s}(l_r).\]

6. **k-primitive functions.**

**Theorem 5.** A function \(f(l_n, l_r)\) is \(k\)-primitive if and only if it can be written in the form

\[(6.1) \quad f(l_n, l_r) = \sum_{d \mid (l_r^k)} \alpha(d, l_r) c_k(l_n, d).\]
PROOF. Suppose first that \( f(l_n, l_r) \) is \( k \)-primitive and so \( k \)-even and therefore the results of (5.1) and (5.2) apply. In this case

\[
d^k|([\gamma(l_n, l_r^k)]^k, l_r^k)_k \iff d^k|[\gamma(l_n, l_r^k)]^k.
\]

Hence

\[
f(l_n, l_r) = \sum_{d^k |([\gamma(l_n, l_r^k)]^k, l_r^k)_k} |\mu(d)| w(d, l_r/d)
\]

and

\[
\alpha(d) = l_r^{-k} \sum_{e |l_r/d} w(l_r/e, e)e^k|\mu(l_r/e)|.
\]

But if \( d \) contains a square factor \( D^2 \), then \( D^2 \) is also a factor of \( l_r/e \) for all divisors \( e \) of \( l_r/d \). Therefore \( \alpha(d) = 0 \) unless \( d \) is primitive and so the necessity half of Theorem 5 is proved. Now suppose \( f(l_n, l_r) \) may be written in the form (6.1). Then

\[
f([\gamma(l_n, l_r^k)]^k, l_r) = \sum_{d |\gamma(l_r)|} \alpha(d) c_k([\gamma(l_n, l_r^k)]^k, d)
\]

\[
= \sum_{d |\gamma(l_r)|} \alpha(d) c_k(l_n, l_r^k, d)
\]

\[
= \sum_{d |\gamma(l_r)|} \alpha(d) c_k(l_n, d).
\]

This completes the proof of Theorem 5.

**Theorem 6.** \( f(l_n, l_r) \) is \( k \)-primitive if and only if it can be written in the form

\[
(6.2) \quad f(l_n, l_r) = \sum_{d |\gamma(l_r)| \atop (l_n, d^k)_k = 1} h(d, g(l_r)/d).
\]

**Proof.** When \( d \) is primitive,

\[
(l_n, d^k)_k = 1 \iff ([\gamma(l_n, l_r^k)]^k, d^k)_k = 1,
\]

and so if \( f(l_n, l_r) \) can be written in the form (6.2),

\[
f(l_n, l_r) = f([\gamma(l_n, l_r^k)]^k, l_r),
\]

and the function is \( k \)-primitive. Now suppose \( f(l_n, l_r) \) is \( k \)-primitive and therefore may be written in the form (6.1). Then use of the result (3.2) in the expression for \( c_k(l_n, d) \) eventually yields (6.2) with

\[
h(d, g(l_r)/d) = d^k \mu(d) \sum_{\delta |d \cap \gamma(l_r) \cap d} \alpha(\delta d) \psi_k(\delta).
\]

This completes the proof of Theorem 6.
7. Nagell’s totient function.

Eckford Cohen in [3] has defined the function \( \varphi^{(s)}(n, r) \), where \( s, n \) and \( r \) are positive integers, as follows: \( \varphi^{(s)}(n, r) \) is the number of solutions \( x_i \mod{r} \), \( x_i, r = 1 \) of the congruence \( n \equiv x_1 + x_2 + \ldots + x_s \mod{r} \). When \( s = 2 \), the function becomes Nagell’s totient function. In [3], Cohen has also shown that

\[
\varphi^{(s)}(n, r) = r^{-1} \sum_{d \mid r} c^s(\delta, r) c(n, d),
\]

where \( c(n, r) \) is Ramanujan’s sum. We therefore now define

\[
\theta_{k, s}(l_n, l_r) = l_r^{-k} \sum_{d \mid l_r} c^s(l_r/d^k, l_r) c_k(l_n, d).
\]

Applying [5, Theorem 6, Corollary] we have

\[
c_k((l_r/d^k, l_r) = \frac{\psi_k(l_r)}{\psi_k(d)} \mu(d).
\]

Hence

\[
\theta_{k, s}(l_n, l_r) = \frac{\psi_k^s(l_r)}{l_r^k} \sum_{d \mid l_r} \frac{\mu^s(d)}{\psi_k^s(d)} c_k(l_n, d).
\]

Because of the Möbius function this may be written

\[
\theta_{k, s}(l_n, l_r) = \frac{\psi_k^s(l_r)}{l_r^k} \sum_{d \mid \gamma(l_r)} \frac{\mu^s(d)}{\psi_k^s(d)} c_k(l_n, d).
\]

From Theorem 5 we can see therefore that \( l_r^{-k} \theta_{k, s}(l_n, l_r)/\psi_k^s(l_r) \) is a \( k \)-primitive function. Here (7.3) gives us the value for \( \alpha(d) \) and applying Theorem 6, with the value for \( h(d, \gamma(l_r)/d) \) given by (6.3) gives

\[
\frac{l_r^k}{\psi_k^s(l_r)} \theta_{k, s}(l_n, l_r) = \sum_{d \mid \gamma(l_r)} \frac{d^k \mu^{s+1}(d)}{\psi_k^s(d)} \sum_{\delta \mid \gamma(l_r)/d} \frac{\mu^s(\delta)}{\psi_k^{s-1}(\delta)}.
\]

But from the definition (2.3), the second sum equals \( \psi_{k, s}(1, \gamma(l_r)/d) \) and so

\[
\theta_{k, s}(l_n, l_r) = \frac{\psi_k^s(l_r)}{l_r^k} \sum_{d \mid \gamma(l_r)} \frac{d^k \mu^{s+1}(d)}{\psi_k^s(d)} \psi_{k, s} \left( 1, \frac{\gamma(l_r)}{d} \right).
\]

This general formula only becomes analogous to Cohen’s evaluation of Nagell’s totient function in the special case \( s = 2 \).

**Theorem 7.**

\[
\theta_{k, s}(l_n, l_r) = l_r^{-k} \sum_{d^k \mid l_m, l_r} d^k c^s((l_r/d^k, l_r) \psi_{k, s+1}(d, l_r).
\]
Proof. Put

\[ F(l_n, l_r) = \frac{l_r^k}{\psi_k^s(l_r)} \theta_{k,s}(l_n, l_r) = \sum_{d|l_r} \frac{\mu^s(d)}{\psi_k^s(d)} c_k(l_n, d), \]

the last equality being due to (7.2). From (5.1) we conclude that \( F(l_n, l_r) \) is \( k \)-even with its \( \alpha(d) \) given by (7.5). Hence inversion of (5.2) in the usual way gives the corresponding value for \( w(d, l_r/d) \). Substitution of this value for \( w(d, l_r/d) \) back in (5.1) gives

\[ \theta_{k,s}(l_n, l_r) = \frac{\psi_k^s(l_r)}{l_r^k} \sum_{d^k | (l_n, l_r)^k} \frac{\mu(l_r/(de)) \mu^s(l_r/e)}{\psi_k^s(l_r/e)} \]

\[ = \frac{\psi_k^s(l_r)}{l_r^k} \sum_{e | l_r/d} \frac{d^k \mu(\Delta) \mu^s(d\Delta)}{\psi_k^s(d\Delta)}. \]

For non-zero terms \( \Delta \) must be primitive and \( d\Delta \) primitive and so \( (d, \Delta) = 1 \) and \( d \) is also primitive. Therefore

\[ \theta_{k,s}(l_n, l_r) = l_r^{-k} \sum_{d^k | l_n} \frac{d^k \psi_k^s(l_r) \mu^s(d)}{\psi_k^s(d) \sum_{\Delta | l_r/d} \mu^{s+1}(\Delta)} \]

\[ = l_r^{-k} \sum_{d^k | (l_n, l_r)^k} d^k c_k^{s}((l_r/d)^k, l_r) \psi_{k,s+1}(1, l_r/d) \]

from [5, Theorem 6, Corollary]. Now

\[ \psi_{k,s+1}(d, l_r) = \sum_{\delta | l_r} \frac{\mu^{s+1}(\delta)}{\psi_k^s(\delta)} \]

and \( d | l_r \) also, and so \( (d, \delta) = 1 \) only if \( \delta | l_r/d \), and in this particular case

\[ \psi_{k,s+1}(d, l_r) = \psi_{k,s+1}(1, l_r/d). \]

This completes the proof.

Again, applying [5, Theorem 7 (the inversion formula)] to (7.1) gives

\[ c_k^s(l_n, l_r) = \sum_{\delta | l_r} \theta_{k,s}((l_r/d)^k, l_r) c_k(l_n, d). \]

Putting \( l_n = 0 \) in (7.6) we obtain

\[ \psi_k^s(l_r) = \sum_{\delta | l_r} \theta_{k,s}((l_r/d)^k, l_r) \psi_k(d). \]

Putting \( l_n = 1 \) in (7.6) we have \( c_k(1, l_r) = \mu(l_r) \) and

\[ \mu^s(l_r) = \sum_{\delta | l_r} \theta_{k,s}((l_r/d)^k, l_r) \mu(d). \]
Another property of \( \theta_k(s; (l_n, l_r)) \) may be obtained by using Theorem 3. Put \( f(l_n, l_r) = c_k(l_n, l_r) \) and \( g(l_n, l_r) = \theta_k(s; (l_n, l_r)) \). Then \( c_k(l_n, l_r) = \sum_{d \mid l_r} \alpha(d, l_r) c_k(l_n, d) \).

Hence in this case

\[
\alpha(d, l_r) = \begin{cases} 0 & \text{if } d < l_r \\ 1 & \text{if } d = l_r . \end{cases}
\]

Also from (7.2),

\[
\beta(d, l_r) = \frac{\psi_k^s(l_r)}{l_r^k} \frac{\mu^s(d)}{\psi_k^s(d)} .
\]

From Theorem 3 we therefore have

\[
\sum_{d = l_r} c_k(e_k, l_r) \theta_k(s; (e_k, l_r)) \psi_k(d)
\]

\[
= l_r^k \frac{\psi_k^s(l_r)}{l_r^k} \frac{\mu^s(l_r)}{\psi_k^s(l_r)} \psi_k(l_r) = \mu^s(l_r) \psi_k(l_r) .
\]

**Special case** \( s = 2 \). Putting \( s = 2 \) in (7.4) we have

\[
\theta_k,2(l_n, l_r) = \frac{\psi_k^2(l_r)}{l_r^k} \sum_{d \mid l_r} \frac{d^k \mu^3(d)}{\psi_k^2(d)} \frac{1, \gamma(l_r)}{d} .
\]

Now \( \psi_k,2(1, l_r) \) is multiplicative in \( l_r \) and we need therefore only evaluate it for the prime power \( p^a \):

\[
\psi_k,2(1, p^a) = \sum_{d \mid p^a} \frac{\mu^2(d)}{\psi_k(d)} = \frac{1}{\psi_k(1)} + \frac{1}{\psi_k(p)} = 1 + \frac{1}{p^k - 1} = \frac{p^k}{\psi_k(p)} .
\]

Hence

\[
\psi_k,2 \left( 1, \frac{\gamma(l_r)}{d} \right) = \frac{\gamma^k(l_r)}{d^k} \frac{\gamma_k(d)}{\psi_k(\gamma(l_r))} .
\]

However it is easily verified from (2.2) that

\[
\psi_k(l_r) = \frac{\psi_k(\gamma(l_r))}{\gamma^k(l_r)} ,
\]

and so

\[
\theta_k,2(l_n, l_r) = \psi_k(l_r) \sum_{d \mid \gamma(l_r)} \frac{\mu(d)}{\psi_k(d)} .
\]

Because of the Möbius function we therefore have

\[
\theta_k,2(l_n, l_r) = \psi_k(l_r) \sum_{d \mid l_r, (l_n, d^k) = 1} \frac{\mu(d)}{\psi_k(d)} .
\]
This formula is analogous to Cohen's evaluation of Nagell's totient function.

8. Another analogue of Nagell's totient function.

P. J. McCarthy in [8] has defined the function $N_k^*(n, r, t)$ to be the number of solutions (mod $r^k$) of the congruence

$$n \equiv x_1 + x_2 + \ldots + x_t \pmod{r^k},$$

where $(x_1, x_2, \ldots, x_t, r^k)_k = 1$, and quoted the result

$$N_k^*(n, r, t) = r^{-k} \sum_{d|r} c_{k,t} (r/d)^{kt}, r, c_k(n, d).$$

When $t = 2$ we have another analogue of Nagell's totient function. We therefore define

$$\theta_{k,s}^*(l, r) = l^{-k} \sum_{d|l} c_{k,s}(l/d)^{ks}, l, c_k(l, d).$$

Applying Theorem 4 to this definition we therefore have

$$\theta_{k,s}^*(l, r) = l^{k(s-1)}(l, r)_k^{1-s} \psi_{ks-k}(l, r)_k^{1/k}. $$

Applying [5, Theorem 6, Corollary], we have

$$c_{ks}(l/d)^{ks}, l, c_k(l, d) = \frac{\psi_{ks}(l, d)}{\psi_{ks}(d)}$$

and so from (8.1)

$$\theta_{k,s}^*(l, r) = \frac{\psi_{ks}(l, r)}{l_r^k} \sum_{d|l} \frac{\mu(d) c_k(l, d)}{\psi_{ks}(d)}$$

$$= \frac{\psi_{ks}(l, r)}{l_r^k} \sum_{d|l} \frac{\mu(d) c_k(l, d)}{\psi_{ks}(d)}.$$

From Theorem 5 it follows that

$$\frac{l_r^k \theta_{k,s}^*(l, r)}{\psi_{ks}(l, r)}$$

is $k$-primitive with $\alpha(d) = \mu(d)/\psi_{ks}(d)$. From Theorem 6 we may therefore write

$$\theta_{k,s}^*(l, r) = \frac{\psi_{ks}(l, r)}{l_r^k} \sum_{d|l} \frac{\mu(d) \sum_{\delta|\delta(d)} \frac{\mu(\delta \delta d)}{\psi_{ks}(\delta \delta)}}{\psi_{ks}(\delta \delta)}.$$

$$= \frac{\psi_{ks}(l, r)}{l_r^k} \sum_{d|l} \frac{\mu(\delta \delta d)}{\psi_{ks}(\delta \delta)}.$$
However from Example 5 in [9] and [5] we have

\[(8.4) \quad \sum_{d|l_r} \frac{\mu(d) \psi_k(d)}{\psi_l(d)} = \frac{l_r^k \psi_{t-k}(l_r)}{\psi_l(l_r)}, \quad t - k \neq 0.\]

Thus

\[
\sum_{d|\gamma(l_r)/d} \frac{\mu(d) \psi_k(d)}{\gamma_k(d)} = \frac{\gamma^k(l_r)}{\gamma_k^k} \frac{\psi_{k\gamma}(\gamma(l_r)/d)}{\psi_{k\gamma}(\gamma(l_r)/d)}.
\]

Hence

\[
\theta_{k,\delta}^*(l_n, l_r) = \psi_{ks-k}(l_r) \frac{\psi_{ks-k}(\gamma(l_r))}{\psi_{ks}(\gamma(l_r))} \sum_{d|\gamma(l_r) \atop (l_n, d') = 1} \frac{\mu^2(d)}{\psi_{ks-k}(d)}
\]

from (7.10). We may therefore write

\[(8.5) \quad \theta_{k,\delta}^*(l_n, l_r) = \psi_{ks-k}(l_r) \sum_{d|l_r \atop (l_n, d') = 1} \frac{\mu^2(d)}{\psi_{ks-k}(d)}.
\]

This general formula is analogous to McCarthy’s evaluation of Nagell’s totient function.

From definition (8.1) it can be seen that \(\theta_{k,\delta}^*(l_n, l_r)\) is \(k\)-even and so the inversion formula, Theorem 7 of [5], may be applied. In (8.1) put \(ks = t\) and

\[f(l_n, l_r) = l_r^k \theta_{k,\delta}^*(l_n, l_r) = \sum_{d|l_r} c_d((l_r/d)^t, l_r) \psi_k(l_n, d)\]

and then \(w(d, l_r) = c_d((l_r/d)^t, l_r)\). Hence

\[w(l_m, l_r) = c_d((l_r/l_m)^t, l_r) \quad \text{and} \quad l_m^k = \frac{l_r^k}{(l_n, l_r^k)}.\]

Thus

\[(l_r/l_m)^t = ((l_r/l_r)(l_n, l_r)^t)^{(l_r)} = (l_n, l_r)^{l_r}.
\]

Hence

\[w(l_m, l_r) = c_d((l_n, l_r^k)^{l_r}, l_r) = c_{ks}(l_n, l_r^k)^{l_r}, l_r) = c_{ks}(l_n, l_r^k),
\]

by the definition of \(c_{ks}\). Hence

\[(8.6) \quad c_{ks}(l_n^k, l_r) = l_r^{-k} \sum_{d|l_r} \psi_{k,\delta}^*((l_r/d)^k, l_r) \psi_{k}(l_n, d)
\]

\[= \sum_{d|l_r} \theta_{k,\delta}^*((l_r/d)^k, l_r) \psi_{k}(l_n, d).\]
In the special case $l_n = 0$ we have

\[(8.7) \quad \psi_{k,s}(l_r) = \sum_{d|l_r} \theta_{k,s}^*((l_r/d)^k, l_r) \psi_k(d).\]

REFERENCES


UNIVERSITY OF NEW ENGLAND, ARMIDALE, N.S.W., AUSTRALIA