A DECOMPOSITION THEOREM FOR $C^*$ ALGEBRAS

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As pointed out to us by T. Bai Andersen the proof of [1, theorem 1.4] is incomplete. We give here a correct proof (corollary 6) as a consequence of a decomposition theorem, which may have independent interest. The observation, as well as the proof, that the decomposition theorem implies the result of E. Størmer [2] is due to F. Combes.

Let $A$ be a $C^*$ algebra represented as operators on some Hilbert space. For any operator $x$ let $[x]$ denote the range projection of $x$.

Let $x_1, x_2, \ldots, x_k$ be a finite set of positive operators from $A$, and set $x = \sum x_i$.

We denote by $B^+$ the smallest closed order ideal in $A^+$ containing $x$. Then [1, Lemma 1.1] $B^+$ is the uniform closure of the order ideal

$$\{y \in A^+ \mid \exists x \in B^+: y \leq \alpha x\},$$

also $B^+$ is the positive part of an order-related $C^*$ subalgebra $B$, where $B = L^* \cap L$, $L$ being the closed left ideal generated by $x$. Clearly $B$ is also the order-related $C^*$ subalgebra generated by $x_1, x_2, \ldots, x_k$.

**Theorem 1.** For any $y \in B^+$ there exist $y_i \in B$, $y_i = [x_i]y_i$, $i = 1, 2, \ldots, k$, such that $y = \sum y_i^*y_i$.

**Proof.** Define $u_n = (n^{-1} + x)^{-1}$. Then $0 \leq u_n \leq u_m \leq 1$ for $n \leq m$. For any $z \in B$ there is a sequence $\{\alpha_m\}$ such that $z^*z \leq \alpha_m x + m^{-1}$. Hence

$$\|z(1 - u_n)\|^2 = \|(1 - u_n)z^*z(1 - u_n)\| \leq m^{-1} + \alpha_m \sup_{\alpha} \frac{n^{-2} \alpha}{(n^{-1} + \alpha)^2} \leq m^{-1} + \alpha_m (4n)^{-1}.$$ 

It follows that $\{u_n\}$ is an approximative unit for $B$. Now take any $y \in B^+$. Then

$$y = \lim y^\dag u_n^2 y^\dag = \lim y^\dag (n^{-1} + x)^{-1} x^\dag (\sum x_i) x^\dag (n^{-1} + x)^{-1} y^\dag = \lim \sum (y^\dag (n^{-1} + x)^{-1} x^\dag x_i^\dag) (x_i^\dag x^\dag (n^{-1} + x)^{-1} y^\dag).$$

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But \( y_{i,n} = x_i^1 x_i^1 (n^{-1} + x)^{-1} y_i \) converges to an element \( y_i \in B \) since

\[
\|y_{i,n} - y_{i,m}\|^2 = \|x_i^1 x_i^1 ((n^{-1} + x)^{-1} - (m^{-1} + x)^{-1}) y_i\|^2 \\
\leq \|y_i^2 ((n^{-1} + x)^{-1} - (m^{-1} + x)^{-1})^2 x_i^1 y_i\|^2 = \|(u_n - u_m) y_i\|^2.
\]

Hence \( y = \sum y_i^* y_i \) with \( y_i = [x_i] y_i \).

**Corollary 2** (Riesz decomposition property). If \( 0 \leq y \leq \sum x_i \), then there exist \( y_i \in B \) such that \( y = \sum y_i^* y_i \), \( y_i y_i^* \leq x_i \).

**Proof.** With \( y_i \) as in the proof of theorem 1, we observe that

\[
y_{i,n} y_i^* = x_i^1 x_i^1 (n^{-1} + x)^{-1} y(n^{-1} + x)^{-1} x_i^1 \leq x_i^1 u_n^2 x_i^1 \leq x_i.
\]

**Proposition 3** (Størmer). If \( I \) and \( J \) are closed two-sided ideals in \( A \), then \( (I + J)^+ = I^+ + J^+ \).

**Proof.** Suppose \( x \in (I + J)^+ \). Then there exist self-adjoint elements \( y \in I, z \in J \) with \( x = y + z \), hence \( x \leq |y| + |z| \). By corollary 2 there exist \( u, v \) such that

\[
x = u^* u + v^* v, \quad uu^* \leq y, \quad vv^* \leq z.
\]

We have \( u^* u \in I^+ \) and \( v^* v \in J^+ \), and proposition 3 follows.

Let \( K \) denote the intersection of all dense, order-related two-sided ideals in \( A \) (see [1, section 1] for a discussion). Then \( K^+ \) is the smallest order ideal containing all elements \( x \in A^+ \) such that there exist elements \( y \in A^+ \) with \( [x] \leq y \). However the smallest order ideal \( J \) containing all elements \( x \in A^+ \), such that there exists \( y \in K^+ \) with \( [x] \leq y \), is also an invariant order ideal, and is dense in \( A^+ \) by the same argument which proves the density of \( K^+ \). Hence \( J = K^+ \).

With the above remarks in mind we may proceed with

**Proposition 4.** If \( \{x_i\} \) is a finite set of elements from \( K \), then the order-related \( C^* \) algebra they generate is also in \( K \).

**Proof.** Each element \( x_i \) is a linear combination of elements from \( K^+ \) each of which is majorized by a finite sum of elements \( y_j \) for which \( [y_j] \leq z_j \) for some \( z_j \in K^+ \).

Hence the \( C^* \) algebra generated by the \( x_i \)'s is contained in the order-related \( C^* \) subalgebra generated by the \( y_j \)'s. But every element in this algebra is spanned by positive elements \( y \) for which, by theorem 1, we have a decomposition.
$$y = \sum v_j^*[y_j]v_j \leq \sum v_j^*z_jv_j \in K^+.$$ Hence $y \in K^+$.

**Proposition 5.** If $\Phi$ is a *homomorphism from a C* algebra $A$ onto a C* algebra $B$, and if $a \in A^+$, $b = \Phi(a)$, then

$$\Phi\{x \in A^+ | x \leq a\} = \{y \in B^+ | y \leq b\}.$$ 

**Proof.** Suppose $y \in B^+$, $y \leq b$. Since $\Phi(A^+) = B^+$, there exists a self-adjoint $z \in A$ such that $z \leq a$, $\Phi(z) = y$. We have $\Phi(z_+) = y$, $\Phi(z_-) = 0$ and $z_+ \leq a + z_-$, hence the sequence with elements

$$u_n = z_+^i(n^{-1} + z_- + a)^{-1}(z_- + a)\alpha^i a^\alpha$$

is convergent (compare with the sequence $\{y_{in}\}$ in the proof of theorem 1). Set $x = \lim u_n^*u_n \leq a$, and observe that

$$\Phi(x) = \lim b(n^{-1} + b)^{-1}y(n^{-1} + b)^{-1}b = [b]y[b] = y.$$ 

**Corollary 6.** $\Phi(K_A) = K_B$.

**Proof.** Since $\Phi$ preserves spectral theory, $\Phi(K_A) \subseteq K_B$. On the other hand $\Phi(K_A)$ is clearly a dense, two-sided ideal in $B$. By proposition 5, $\Phi(K_A^+)$ is an order ideal, hence $\Phi(K_A)$ is order-related. Since $K_B$ is minimal among all dense, order-related, two-sided ideals in $B$, we conclude $K_B \subseteq \Phi(K_A)$.

**References**


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