ON THE ZEROS OF AN ENTIRE FUNCTION III

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Let $f(z)$ be an entire function of order $\varrho$ and lower order $\lambda$ such that $0 < \lambda \leq \varrho < \infty$. Let $\varrho_1$ and $\lambda_1$ denote the exponent of convergence and the lower exponent of convergence of the zeros of $f(z)$, respectively.

It is known that corresponding to every entire function of order $\varrho$, $0 < \varrho < \infty$, there exists a function $\varrho(r)$ called its proximate order having the following properties:

i) $\varrho(r)$ is real, continuous and piecewise differentiable for $r \geq r_0$.
ii) $\varrho(r) \to \varrho$ as $r \to \infty$.
iii) $r \varrho'(r) \log r \to 0$ as $r \to \infty$.
iv) $\log M(r) \leq r\varrho(r)$ for $r \geq r_0$,
    $\log M(r) = r\varrho(r)$ for a sequence of values of $r$ tending to $\infty$.

S. M. Shah [3] has proved the existence of a function $\lambda(r)$, for an entire function of finite lower order $\lambda$, analogous to $\varrho(r)$ having the following properties:

i)' $\lambda(r)$ is a non-negative, continuous function of $r$ for $r \geq r_0$, differentiable except at isolated points at which $\lambda'(r - 0)$ and $\lambda'(r + 0)$ exist.
ii)' $\lambda(r) \to \lambda$ as $r \to \infty$.
iii)' $r \lambda'(r) \log r \to 0$ as $r \to \infty$.
iv)' $\log M(r) \geq r\lambda(r)$ for $r \geq r_0$,
    $\log M(r) = r\lambda(r)$ for a sequence of values of $r$ tending to $\infty$.

Let $\varrho_1(r)$ and $\lambda_1(r)$ be the proximate order and the lower proximate order with respect to $n(r, 1/f)$, the number of zeros of $f(z)$ in the circle $|z| \leq r$. Then $\varrho_1(r)$ and $\lambda_1(r)$ have the properties analogous to those of $\varrho(r)$ and $\lambda(r)$, respectively.

We shall prove the following Theorems, where

$$N(r, 1/f) = \int_{r_0}^{r} x^{-1} n(x, 1/f) \, dx.$$
Theorem 1. Let \( f(z) \) be an entire function of finite order \( q \). Let
\[
\limsup_{r \to \infty} \frac{n(r,1/f)}{N(r,1/f)} = S \text{ where } S < \infty,
\]
\[
\liminf_{r \to \infty} \frac{n(r,1/f)}{N(r,1/f)} = T \text{ where } 0 < T.
\]
Then
\[
\frac{T^2}{S} \leq \lambda_1 \leq \varrho_1 \leq \frac{S^2}{T}.
\]

Theorem 2.
\[
\liminf_{r \to \infty} \frac{n(r,1/f)}{r^\lambda_1(r)} \geq \frac{T^{p+2} \log k_1 \log k_2 \ldots \log k_p}{\lambda_1(k_1 k_2 \ldots k_p)^{\lambda_1}},
\]
where \( p \) is a positive integer, \( k_1, k_2, \ldots, k_p \) are constants \( > 1 \), and \( \lambda_1(r) \) is the lower proximate order with respect to \( N(r,1/f) \).

Corollary.
\[
S \geq \frac{T^{p+2} \log k_1 \log k_2 \ldots \log k_p}{\lambda_1(k_1 k_2 \ldots k_p)^{\lambda_1}}.
\]

\( p = 0 \) gives the first part of Theorem 1.

Theorem 3.
\[
\limsup_{r \to \infty} \frac{n(r,1/f)}{r^\varrho_1(r)(\log r)^p} \leq \frac{S^{p+2}}{\varrho_1},
\]
where \( \varrho_1(r) \) is the proximate order with respect to \( N(r,1/f) \).

Corollary.
\[
\liminf_{r \to \infty} \frac{n(r,1/f)}{N(r,1/f)(\log r)^p} \leq \frac{S^{p+2}}{\varrho_1}.
\]

Theorem 4. Let \( a \) and \( b, a \neq b, \) be finite numbers and
\[
\limsup_{r \to \infty} \frac{N(r,1/(f-a)) + N(r,1/(f-b))}{\log M(r,f)} = \alpha,
\]
\[
\liminf_{r \to \infty} \frac{N(r,1/(f-a)) + N(r,1/(f-b))}{\log M(r,f)} = \beta.
\]
Let further \( \varrho(r) \) be the proximate order and \( \lambda(r) \) the lower proximate order with respect to \( \log M(r,f) \). Then the following statements hold:
\[(1) \quad \lim_{r \to \infty} \frac{n(r, 1/(f-a))}{\rho(r)} = 0 \implies \limsup_{r \to \infty} \frac{n(r, 1/(f-b))}{\rho(r)} \geq \beta \varepsilon.
\]

\[(2) \quad \lim_{r \to \infty} \frac{N(r, 1/(f-a))}{\lambda(r)} = 0 \implies \limsup_{r \to \infty} \frac{n(r, 1/(f-b))}{\lambda(r)} \geq \alpha \lambda.
\]

\[(3) \quad \liminf_{r \to \infty} \frac{N(r, 1/(f-a))}{\lambda(r)} = 0 \implies \liminf_{r \to \infty} \frac{n(r, 1/(f-b))}{\lambda(r)} \leq \alpha \lambda.
\]

\[(4) \quad \liminf_{r \to \infty} \frac{N(r, 1/(f-a))}{\rho(r)} = 0 \implies \liminf_{r \to \infty} \frac{n(r, 1/(f-b))}{\rho(r)} \leq \beta \varepsilon.
\]

**Remark.** Statement (2) is an improvement of a result of V. Srinivasulu [4], namely of (2) with $\beta \lambda$ instead of $\alpha \lambda$ on the right-hand side.

**Corollary.** Let

\[
\limsup_{r \to \infty} \frac{n(r, 1/(f-b)) \log r}{N(r, 1/(f-b)) \log N(r, 1/(f-b))} = J
\]

\[
\liminf_{r \to \infty} \frac{n(r, 1/(f-b)) \log r}{N(r, 1/(f-b)) \log N(r, 1/(f-b))} = K.
\]

Then

\[(1)' \quad \lim_{r \to \infty} \frac{N(r, 1/(f-a))}{\lambda(r)} = 0 \implies \limsup_{r \to \infty} \frac{n(r, 1/(f-b))}{\lambda(r)} \geq K \alpha \lambda_1(b),
\]

\[(2)' \quad \lim_{r \to \infty} \frac{N(r, 1/(f-a))}{\rho(r)} = 0 \implies \limsup_{r \to \infty} \frac{n(r, 1/(f-b))}{\rho(r)} \geq K \beta \lambda_1(b),
\]

\[(3)' \quad \liminf_{r \to \infty} \frac{N(r, 1/(f-a))}{\rho(r)} = 0 \implies \liminf_{r \to \infty} \frac{n(r, 1/(f-b))}{\rho(r)} \leq J \beta \varepsilon_1(b),
\]

\[(4)' \quad \liminf_{r \to \infty} \frac{N(r, 1/(f-a))}{\lambda(r)} = 0 \implies \liminf_{r \to \infty} \frac{n(r, 1/(f-b))}{\lambda(r)} \leq J \alpha \varepsilon_1(b).
\]

**Proof of Theorem 1.** We have [4]

\[n(r, 1/f) r^{-\lambda_1(r)} > o(1) + \lambda_1^{-1}(T - \varepsilon)^2,
\]

where $\lambda_1(r)$ is the lower proximate order with respect to $N(r, 1/f)$. Thus,

\[n(r, 1/f) / N(r, 1/f) > \lambda_1^{-1}(T - \varepsilon)^2
\]

for a sequence of values of $r$ tending to $\infty$. Hence

\[
\limsup_{r \to \infty} n(r, 1/f) / N(r, 1/f) \geq \lambda_1^{-1} T^2,
\]

that is, $T^2 / S \leq \lambda_1$. Similarly we get $\varepsilon_1 \leq S^2 / T$. 

Proof of Theorem 2. To $\varepsilon > 0$ there is an $r_0 \geq 0$ such that

$$n(r, 1/f) > (T - \varepsilon) N(r, 1/f) \quad \text{for } r \geq r_0.$$ 

Now, writing $n(x)$ for $n(x, 1/f)$, we have, with $k_1 > 1$,

$$N(r, 1/f) = \int_{r_0}^{r} \frac{n(x)}{x} \, dx > \int_{r/k_1}^{r} \frac{n(x)}{x} \, dx > n(r/k_1) \log k_1,$$

and thus, with $k_2 > 1, \ldots, k_p > 1$,

$$n(r) > (T - \varepsilon) N(r) > (T - \varepsilon) n(r/k_1) \log k_1$$

$$> (T - \varepsilon)^2 n(r/(k_1 k_2)) \log k_1 \log k_2$$

$$\vdots$$

$$> (T - \varepsilon)^p n(r/(k_1 k_2 \ldots k_p)) \log k_1 \log k_2 \ldots \log k_p$$

$$> (T - \varepsilon)^{p+1} \int_{r_0}^{r} n(x) x^{-1} \, dx \log k_1 \log k_2 \ldots \log k_p,$$

$$> (T - \varepsilon)^{p+1} \int_{r_0}^{r} n(x) x^{-1} \, dx \log k_1 \log k_2 \ldots \log k_p.$$

Letting $\lambda_1(r)$ be the lower proximate order with respect to $N(r, 1/f)$ we have

$$n(r) > (T - \varepsilon)^{p+2} \log k_1 \log k_2 \ldots \log k_p \int_{r_0}^{r} x^{\lambda_1(x)^{-1}} \, dx.$$

Hence [2, p. 58]

$$n(r) > \lambda_1^{-1} (T - \varepsilon)^{p+2} \log k_1 \log k_2 \ldots \log k_p \left( r/(k_1 k_2 \ldots k_p) \right)^{\lambda_1(r/(k_1 k_2 \ldots k_p))}$$

$$> \lambda_1^{-1} (T - \varepsilon)^{p+2} \log k_1 \log k_2 \ldots \log k_p \left( r/(k_1 k_2 \ldots k_p) \right)^{-\lambda_1},$$

and thus

$$\liminf_{r \to \infty} n(r, 1/f) \lambda_1(r)^{-1} \geq \lambda_1^{-1} T^{p+2} \log k_1 \log k_2 \ldots \log k_p (k_1 k_2 \ldots k_p)^{-\lambda_1}.$$

Proof of Theorem 3. For $r \geq r_0$ we have

$$n(r, 1/f) < (S + \varepsilon) N(r, 1/f)$$

$$< (S + \varepsilon) n(r, 1/f) \log r$$

$$< (S + \varepsilon)^p n(r) (\log r)^p$$

$$< (S + \varepsilon)^{p+1} N(r) (\log r)^p$$

$$< (S + \varepsilon)^{p+1} (\log r)^p \int_{r_0}^{r} n(x) x^{-1} \, dx.$$
Letting \( \varrho_1(r) \) be the proximate order with respect to \( N(r, 1/f) \) we have

\[
n(r, 1/f) < (S + \varepsilon)^{p+2} (\log r)^p \int_{r_0}^{r} N(x)x^{-1} \, dx
\]

\[
< (S + \varepsilon)^{p+2} (\log r)^p \int_{r_0}^{r} x^{\varrho_1(x)-1} \, dx ,
\]

and thus [2, p. 58]

\[
n(r, 1/f) < (S + \varepsilon)^{p+2} (\log r)^p \frac{r^{\varrho_1(r)}}{\varrho_1}.
\]

Hence

\[
\limsup_{r \to \infty} n(r, 1/f)(\log r)^{-p} r^{-\varrho_1(r)} \leq \frac{S^{p+2}}{\varrho_1}.
\]

and

\[
\liminf_{r \to \infty} n(r, 1/f)(\log r)^{-p} N(r, 1/f) \leq \frac{S^{p+2}}{\varrho_1}.
\]

**Proof of Theorem 4.**

(1) Since

\[
\liminf_{r \to \infty} \frac{N(r, 1/(f-a)) + N(r, 1/(f-b))}{\log M(r, f)} = \beta ,
\]

to each \( \varepsilon > 0 \) there is an \( r_0 \) such that

\[
\{N(r, 1/(f-a)) + N(r, 1/(f-b))\}/\log M(r, f) > \beta - \varepsilon \quad \text{for} \quad r \geq r_0.
\]

Further, there is a sequence of values of \( r \) tending to \( \infty \) for which

\[
\{N(r, 1/(f-a)) + N(r, 1/(f-b))\} r^{-\varrho(r)} > \beta - \varepsilon ,
\]

where \( \varrho(r) \) is the proximate order with respect to \( \log M(r, f) \). Hence

\[
\limsup_{r \to \infty} \{N(r, 1/(f-a)) + N(r, 1/(f-b))\} r^{-\varrho(r)} \geq \beta .
\]

Now,

\[
\limsup_{r \to \infty} N(r, 1/(f-a)) r^{-\varrho(r)} + \limsup_{r \to \infty} N(r, 1/(f-b)) r^{-\varrho(r)}
\]

\[
\geq \limsup_{r \to \infty} \{N(r, 1/(f-a)) r^{-\varrho(r)} + N(r, 1/(f-b)) r^{-\varrho(r)}\}
\]

and as

\[
\lim_{r \to \infty} N(r, 1/(f-a)) r^{-\varrho(r)} = 0 ,
\]

we obtain

\[
\limsup_{r \to \infty} N(r, 1/(f-b)) r^{-\varrho(r)} \geq \beta \Rightarrow \limsup_{r \to \infty} n(r, 1/(f-b)) r^{-\varrho(r)} \geq \beta \varrho .
\]
(2) Since
\[
\limsup_{r \to \infty} \frac{N(r, 1/(f-a)) + N(r, 1/(f-b))}{\log M(r)} = \alpha ,
\]
there is a sequence of values of \( r \) tending to \( \infty \) for which
\[
\frac{N(r, 1/(f-a)) + N(r, 1/(f-b))}{\log M(r)} > \alpha - \varepsilon .
\]
Now, let \( \lambda(r) \) be the lower proximate order with respect to \( \log M(r) \).
Then we have
\[
\{N(r, 1/(f-a)) + N(r, 1/(f-b))\}r^{-\lambda(r)} > \alpha - \varepsilon ,
\]
and as
\[
\lim_{r \to \infty} N(r, 1/(f-a))r^{-\lambda(r)} = 0 ,
\]
we get
\[
\limsup_{r \to \infty} N(r, 1/(f-b))r^{-\lambda(r)} \geq \alpha \implies \limsup_{r \to \infty} n(r, 1/(f-b))r^{-\lambda(r)} \geq \alpha \lambda .
\]
(3) To each \( \varepsilon > 0 \) there is an \( r_0 \) such that
\[
\{N(r, 1/(f-a)) + N(r, 1/(f-b))\}/\log M(r) < \alpha + \varepsilon \quad \text{for} \quad r \geq r_0 ,
\]
and there is a sequence of values of \( r \) tending to \( \infty \) for which
\[
\{N(r, 1/(f-a)) + N(r, 1/(f-b))\}r^{-\lambda(r)} < \alpha + \varepsilon ,
\]
where \( \lambda(r) \) is the lower proximate order with respect to \( \log M(r,f) \). Thus,
\[
\liminf_{r \to \infty} \{N(r, 1/(f-a)) + N(r, 1/(f-b))\}r^{-\lambda(r)} \leq \alpha .
\]
Now,
\[
\liminf_{r \to \infty} N(r, 1/(f-a))r^{-\lambda(r)} + \liminf_{r \to \infty} N(r, 1/(f-b))r^{-\lambda(r)}
\]
\[
\leq \liminf_{r \to \infty} \{N(r, 1/(f-a)) + N(r, 1/(f-b))\}r^{-\lambda(r)} ,
\]
and as
\[
\liminf_{r \to \infty} N(r, 1/(f-a))r^{-\lambda(r)} = 0 ,
\]
we get
\[
\liminf_{r \to \infty} N(r, 1/(f-b))r^{-\lambda(r)} \leq \alpha \implies \liminf_{r \to \infty} n(r, 1/(f-b))r^{-\lambda(r)} \leq \alpha \lambda .
\]
The proof of (4) is similar.
Proof of Corollary. (1)' Since
\[
\limsup_{r \to \infty} \frac{n(r, 1/(f-b))}{r^{\lambda(r)}} \limsup_{r \to \infty} \frac{\log r}{\log N(r, 1/(f-b))} \cdot \limsup_{r \to \infty} \frac{N(r, 1/(f-b)) \log N(r, 1/(f-b))}{n(r, 1/(f-b)) \log r} \geq \limsup_{r \to \infty} \frac{N(r, 1/(f-b))}{r^{\lambda(r)}},
\]
we have
\[
\limsup_{r \to \infty} n(r, 1/(f-b)) r^{-\lambda(r)} \geq \limsup_{r \to \infty} N(r, 1/(f-b)) r^{-\lambda(r)} K \lambda_1(b),
\]
and the statement is a consequence of (2) in Theorem 4.

Similarly (2)' follows.

(3)' Since
\[
\liminf_{r \to \infty} \frac{n(r, 1/(f-b))}{r^{\rho(r)}} \liminf_{r \to \infty} \frac{\log r}{\log N(r, 1/(f-b))} \cdot \liminf_{r \to \infty} \frac{N(r, 1/(f-b)) \log N(r, 1/(f-b))}{n(r, 1/(f-b)) \log r} \leq \liminf_{r \to \infty} \frac{N(r, 1/(f-b))}{r^{\rho(r)}},
\]
we have
\[
\liminf_{r \to \infty} n(r, 1/(f-b)) r^{-\rho(r)} \leq \liminf_{r \to \infty} N(r, 1/(f-b)) r^{-\rho(r)} J \rho_1(b),
\]
and the statement is a consequence of (4) in Theorem 4.

Similarly (4)' follows.

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REFERENCES