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DISCRETENESS OF THE SPECTRUM OF A SECOND ORDER ELLIPTIC DIFFERENTIAL OPERATOR IN $L^2(\mathsf{R}^n)$

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Introduction.

It is classical that the Schrödinger operator $-\Delta+q(x)$ has discrete spectrum if $q(x)$ is bounded below and $q(x) \to +\infty$ as $|x| \to \infty$. In [3] Molčanov gave a necessary and sufficient condition for discrete spectrum under the assumption that $q(x)$ is bounded below. In the present paper we consider potential functions which are not necessarily bounded below. More precisely, our potential is obtained from a function $q(x)$, bounded below with $q(x) \to +\infty$ as $|x| \to \infty$, by addition of a term $v(x)$, which is bounded relative to the operator $-\Delta+q(x)$. The function $v(x)$ may have singularities, for example of the attractive Coulomb type, with charges limited by the rate of growth of $q(x)$. Actually, the operator $-\Delta+q(x)$ is replaced by a more general elliptic operator with discrete spectrum, as described in Section 2. In Lemma 2 we prove the discreteness of such an operator by a method communicated by Professor K. Jörgens. We make use of an abstract perturbation criterion for discrete spectrum, based on comparison of quadratic forms, given in Section 1, Lemma 1.2. Finally, Section 3 contains the main result. The crucial point is the relative boundedness of the perturbation term, proved in Lemma 3. The conclusion is stated in the Theorem (p. 49). We notice that certain first order terms may be added without disturbing the discreteness (Remark 1) and give an example of an operator with discrete spectrum not covered by previous criterions.

For ordinary differential operators of the type $-d^2/dx^2+q(x)$ a stronger result was obtained by I. Brinck [1].

1. A perturbation criterion for discrete spectrum.

It was shown by Friedrichs [2] that there is a one-to-one correspondence between self-adjoint operators, bounded from below, in a Hilbert space $\mathfrak{H}$, and closed quadratic forms, bounded from below, in $\mathfrak{H}$.

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If A is a self-adjoint operator, bounded from below, with domain $D(A)$, then the corresponding form $\boldsymbol{A}$ with domain $D[\boldsymbol{A}]$ is the closure of the form (Au,u) with domain $D(A)$. Conversely, A is obtained from $\boldsymbol{A}$ by setting

$$D(A) = \{x \in D[\boldsymbol{A}] \mid \exists\, y \in \mathfrak{H}\colon\ \boldsymbol{A}[x,z]=(y,z)\ \forall z \in D[\boldsymbol{A}]\}$$

and

$$Ax = y \quad \text{for } x \in D(A)\,.$$

If $A^{\frac{1}{2}}$ denotes the positive square-root of the positive definite operator A, we have

$$D[\boldsymbol{A}] = D(A^{\frac{1}{2}})\,,$$

and

$$\boldsymbol{A}[u,v] = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v) \quad \text{for } u,v \in D[\boldsymbol{A}]\,.$$

We shall work primarily with the quadratic form $\boldsymbol{A}$ and define A from $\boldsymbol{A}$ as above.

The spectrum of A is denoted by $\sigma(A)$.

LEMMA 1.1. *Let A be a positive definite, self-adjoint operator in $\mathfrak{H}$. Then $\sigma(A)$ is discrete if and only if $A^{-\frac{1}{2}}$ is compact.*

PROOF. We refer to [4, § 24,5, Satz 11].

LEMMA 1.2. *Let A and B be self-adjoint operators in $\mathfrak{H}$, bounded from below, with the corresponding closed quadratic forms $\boldsymbol{A}$ and $\boldsymbol{B}$. If $\sigma(A)$ is discrete, and $D[\boldsymbol{A}]=D[\boldsymbol{B}]$, then $\sigma(B)$ is discrete.*

PROOF. We can assume, that A and B are positive definite. If $\sigma(A)$ is discrete, then by Lemma 1.1 the operator $A^{-\frac{1}{2}}$ is compact. Since A and B are closed on the same domain, the $\boldsymbol{A}$- and $\boldsymbol{B}$-metrics are equivalent, hence the operator $A^{\frac{1}{2}}B^{-\frac{1}{2}}$ is bounded. It follows that the operator

$$B^{-\frac{1}{2}} = A^{-\frac{1}{2}}\, A^{\frac{1}{2}}B^{-\frac{1}{2}}$$

is compact, and $\sigma(B)$ is discrete by Lemma 1.1.

2. The unperturbed operator.

All functions considered are complex-valued functions on real n-space R^n. Points of R^n are denoted $x=(x_1,x_2,\ldots,x_n)$ or $t=(t_1,t_2,\ldots,t_n)$.

Let $\mathscr{M}$ be the formal differential operator defined by

$$\mathscr{M} = -\sum_{i,j=1}^{n} \partial_i a_{ij} \partial_j + q\,,$$

where q and the coefficients a_{ij} are functions from R^n to C satisfying the conditions

$$\text{(i)} \qquad \begin{cases} a_{ij} = \bar{a}_{ji}, \quad a_{ij} \in L^1_{\text{loc}}, \\ \sum_{i,j=1}^{n} a_{ij}(x)\, \bar{\xi}_i \xi_j \geqq a(x)\,|\xi|^2 \quad \text{for } \xi \in \mathsf{C}^n,\ x \in \mathsf{R}^n, \end{cases}$$

where

$$a(x) \geqq c(R) > 0 \quad \text{for } |x| \leqq R,\ 0 \leqq R < \infty,$$

c some function, and

$$\text{(ii)} \qquad \begin{cases} q(x) \geqq 1 \text{ for all } x \in \mathsf{R}^n, \quad q \in L^1_{\text{loc}}, \\ q(x) \to +\infty \text{ as } |x| \to \infty. \end{cases}$$

Let $\boldsymbol{M}_0$ be the quadratic form with domain C_0^1 defined by

$$\boldsymbol{M}_0[u] = \int_{\mathsf{R}^n} \left(\sum_{i,j=1}^{n} a_{ij} \partial_i \bar{u} \partial_j u + q\,|u|^2 \right), \quad u \in C_0^1.$$

$\boldsymbol{M}_0$ is bounded below and densely defined in L^2, so $\boldsymbol{M}_0$ admits a closure $\boldsymbol{M}$ in L^2. The self-adjoint operator, bounded below, corresponding to $\boldsymbol{M}$, is denoted by M.

LEMMA 2. *Under the assumptions* (i) *and* (ii), *the operator M has discrete spectrum.*

PROOF. Assume that $\sigma(M)$ is not discrete. Then there exists a real number λ and a sequence $\{u_\nu\}$ of functions $u_\nu \in D(M)$, such that

$$\|u_\nu\| = 1, \qquad \|Mu_\nu - \lambda u_\nu\| \to 0 \quad \text{as } \nu \to \infty,$$

$$\int_{|x| \leqq R} |u_\nu|^2 \to 0 \quad \text{as } \nu \to \infty \qquad \text{for every } R > 0.$$

Then there exists a real number K such that

$$\left| \int_{\mathsf{R}^n} \left(\sum a_{ij}(x) \partial_i \bar{u}_\nu \partial_j u_\nu + (q(x) - \lambda)\,|u_\nu(x)|^2 \right) dx \right| < K,$$

and hence

$$\int_{\mathsf{R}^n} q\,|u_\nu|^2 < K \quad \text{for all } \nu.$$

Since $q(x) \to +\infty$ as $|x| \to \infty$, there exists for any $\varepsilon > 0$ a number $R_0 > 0$ such that

$$\int_{|x| \geqq R_0} |u_\nu|^2 < \varepsilon \quad \text{for all } \nu.$$

Also, for ν large enough,

$$\int\limits_{|x|\leqq R_0} |u_\nu|^2 < \varepsilon ,$$

and hence $\|u_\nu\|^2 < 2\varepsilon$, contradicting $\|u_\nu\| = 1$.

3. The perturbed operator.

We denote by S_n the unit sphere in R^n and by Γ_n the area of S_n. Let r be a function from R^n to R, such that

(i) $0 < r(x) \leqq 1, \ x \in \mathsf{R}^n$,
(ii) $\inf_{|x|=R} r(x) \geqq c(R) > 0$ for $R > 0$,
(iii) $r(s_1\xi) \geqq r(s_2\xi)$ for $\xi \in S_n, \ 0 \leqq s_1 \leqq s_2$,
(iv) $\sup_t |r(t) - r(x)|/r(t) \to 0$ as $|x| \to \infty$,

where the supremum is taken over all t belonging to the ball

$$S_{x,r} = \{t \mid |t-x| \leqq r(x)\} ,$$

Let β_i, $i = 1,2$, be fixed, $-1 < \beta_i < 1$. Let $v(x)$ be a real-valued function in L^1_{loc} and set for $i = 1,2$

$$V_i(x) = \int\limits_{S_{x,r}} |v(t)| \, |t-x|^{1-n+\beta_i} \, dt .$$

Suppose that the functions $V_i(x)$ satisfy the conditions

(v) ess $\sup_{|x|\leqq R} V_i(x) < \infty$ for $R > 0, \ i = 1,2$.
(vi) $\lim_{R\to\infty} \left\{ \text{ess sup}_{|x|\geqq R} \dfrac{V_1(x) r(x)^{-1-\beta_1}}{q(x)\Gamma_n(1-\beta_1)} + \text{ess sup}_{|x|\geqq R} \dfrac{V_2(x) r(x)^{1-\beta_2}}{a(x)\Gamma_n(1-\beta_2)} \right\} < 1.$

The formal differential operator $\mathscr{L}$ is defined by

$$\mathscr{L} = \mathscr{M} + v = -\sum_{i,j=1}^{n} \partial_i a_{ij} \partial_j + q + v .$$

The quadratic form $\boldsymbol{L}_0$ is defined for $u \in C_0^1$ by

$$\boldsymbol{L}_0[u] = \int\limits_{\mathsf{R}^n} \left(\sum_{i,j=1}^{n} a_{ij} \, \partial_i \bar{u} \, \partial_j u + (q+v)|u|^2 \right) .$$

LEMMA 3. *Under the assumptions* (i)–(vi) *there exist* $\mu < 1$ *and* $K > 0$, *such that for* $u \in C_0^1$

$$\int\limits_{\mathsf{R}^n} v|u|^2 \leqq \mu \boldsymbol{M}[u] + K\|u\|^2 .$$

Proof. For $0<\varepsilon<1$ we denote by r_ε the function defined by

$$r_\varepsilon(x) = (1-\varepsilon)r(x), \quad x \in \mathsf{R}^n .$$

For $u \in C_0^1$, $\xi \in S_n$ and ϱ a real parameter, we have for all $x \in \mathsf{R}^n$

$$(1) \quad u(x) = -\frac{1}{r_\varepsilon(x)} \int_0^{r_\varepsilon(x)} \frac{d}{d\varrho}[(r_\varepsilon(x)-\varrho)u(x+\varrho\xi)]\,d\varrho$$

$$= \frac{1}{r_\varepsilon(x)} \int_0^{r_\varepsilon(x)} u(x+\varrho\xi)\,d\varrho - \frac{1}{r_\varepsilon(x)} \int_0^{r_\varepsilon(x)} (r_\varepsilon(x)-\varrho)\frac{d}{d\varrho}u(x+\varrho\xi)\,d\varrho$$

Hence

$$(2) \quad |u(x)| \leqq \frac{1}{r_\varepsilon(x)} \int_0^{r_\varepsilon(x)} |u(x+\varrho\xi)|\,d\varrho + \int_0^{r_\varepsilon(x)} |\nabla u(x+\varrho\xi)|\,d\varrho, \qquad x \in \mathsf{R}^n .$$

By Schwarz' inequality we have for all $C>0$

$$(3) \quad |u(x)|^2 \leqq \frac{1+C}{1-\beta_1} r_{\varepsilon(x)}^{-1-\beta_1} \int_0^{r_\varepsilon(x)} |u(x+\varrho\xi)|^2 \varrho^{\beta_1}\,d\varrho + $$
$$+ \frac{1+C^{-1}}{1-\beta_2} r_{\varepsilon(x)}^{1-\beta_2} \int_0^{r_\varepsilon(x)} |\nabla u(x+\varrho\xi)|^2 \varrho^{\beta_2}\,d\varrho .$$

Integrating with respect to ξ over S_n, we get

$$(4) \quad |u(x)|^2 \leqq \frac{1+C}{\Gamma_n(1-\beta_1)} r_{\varepsilon(x)}^{-1-\beta_1} \int_{S_{x,r_\varepsilon}} |u(t)|^2\,|t-x|^{1-n+\beta_1}\,dt +$$
$$+ \frac{1+C^{-1}}{1-\beta_2} r_{\varepsilon(x)}^{1-\beta_2} \int_{S_{x,r_\varepsilon}} |\nabla u(t)|^2\,|t-x|^{1-n+\beta_2}\,dt .$$

Multiplication of (4) by $|v(x)|$, integration with respect to x over R^n and interchange of the order of integration gives

$$(5) \quad \int_{\mathsf{R}^n} |v(x)|\,|u(x)|^2\,dx$$
$$\leqq \frac{(1+C)(1-\varepsilon)^{-1-\beta_1}}{\Gamma_n(1-\beta_1)} \int_{\mathsf{R}^n} \left\{ \int_{\Sigma_{t,r_\varepsilon}} r(x)^{-1-\beta_1}\,|v(x)|\,|x-t|^{1-n+\beta_1}\,dx \right\} |u(t)|^2\,dt +$$
$$+ \frac{(1+C^{-1})(1-\varepsilon)^{1-\beta_2}}{\Gamma_n(1-\beta_2)} \int_{\mathsf{R}^n} \left\{ \int_{\Sigma_{t,r_\varepsilon}} r(x)^{1-\beta_2}\,|v(x)|\,|x-t|^{1-n+\beta_2}\,dx \right\} |\nabla u(t)|^2\,dt ,$$

where

$$\Sigma_{t,r_\varepsilon} = \{x \mid |t-x| \leqq r_\varepsilon(x)\}\,.$$

By (vi) there exist $C, R_0 > 0$ and $\varkappa < 1$ such that

$$\frac{1+C}{\Gamma_n(1-\beta_1)} \operatorname{ess\,sup}_{|t| \geqq R_0} \left\{\frac{V_1(t)}{q(t)}\, r(t)^{-1-\beta_1}\right\} \leqq \varkappa \tag{6}$$

and

$$\frac{1+C^{-1}}{\Gamma_n(1-\beta_2)} \operatorname{ess\,sup}_{|t| \geqq R_0} \left\{\frac{V_2(t)}{a(t)}\, r(t)^{1-\beta_2}\right\} \leqq \varkappa\,. \tag{7}$$

Set $\mu = \frac{1}{2}(1+\varkappa)$ and fix ε such that

$$\frac{(1+C)(1-\varepsilon)^{-2-2\beta_1}}{\Gamma_n(1-\beta_1)} \operatorname{ess\,sup}_{|t| \geqq R_0} \left\{\frac{V_1(t)}{q(t)}\, r(t)^{-1-\beta_1}\right\} \leqq \mu\,. \tag{8}$$

By (iv) there exists $R(\varepsilon)$ such that for $|x| \geqq R(\varepsilon)$ and $|t-x| \leqq r(x)$

$$(1-\varepsilon) r(t) \leqq r(x) \leqq (1-\varepsilon)^{-1} r(t)\,. \tag{9}$$

Set

$$R_1 = \max\{R_0, R(\varepsilon)\}, \qquad R_2 = R_1 + \varepsilon^{-1}\,.$$

Let $0 < \delta \leqq c(R_2)$ (cf. (ii)) and define a function s by

$$s(x) = \begin{cases} \delta & \text{for} \quad |x| \leqq R_1 \\ \delta + \varepsilon(|x| - R_1)\big(r(R_2 x/|x|) - \delta\big) & \text{for} \quad R_1 < |x| \leqq R_2\,, \\ r(x) & \text{for} \quad R_2 < |x|\,. \end{cases}$$

Replace the function r by the function s in (1)–(5). It is easy to see, that s satisfies (9). Then we obtain the following estimates of the terms on the right hand side of (5).

By (9) and (8) we get, since $s(t) = r(t)$ for $|t| \geqq R_2$,

$$\begin{aligned} &\frac{(1+C)(1-\varepsilon)^{-1-\beta_1}}{\Gamma_n(1-\beta_1)} \int\limits_{|t| \geqq R_2} \left\{ \int\limits_{\Sigma_{t,s_\varepsilon}} s(x)^{-1-\beta_1} |v(x)|\, |x-t|^{1-n+\beta_1}\, dx \right\} |u(t)|^2\, dt \qquad (10) \\ &\leqq \frac{(1+C)(1-\varepsilon)^{-2-2\beta_1}}{\Gamma_n(1-\beta_1)} \int\limits_{|t| \geqq R_2} \left\{ r(t)^{-1-\beta_1} \frac{V_1(t)}{q(t)} \right\} q(t)\, |u(t)|^2\, dt \\ &\leqq \mu \int\limits_{|t| \geqq R_2} q(t)\, |u(t)|^2\, dt\,, \end{aligned}$$

where $\Sigma_{t,s_\varepsilon} = \{x \mid |t-x| \leqq s_\varepsilon(x)\}$. By (iii), the inequality $s(t) \leqq r(t)$ holds for all $t \in \mathsf{R}^n$, and we obtain from (9) and (7)

$$(11)\quad \frac{(1+C^{-1})(1-\varepsilon)^{1-\beta_2}}{\Gamma_n(1-\beta_2)} \int\limits_{|t|\geqq R_1} \left\{ \int\limits_{\Sigma_{t,s_\varepsilon}} s(x)^{1-\beta_2}\, |v(x)|\; |x-t|^{1-n+\beta_2}\, dx \right\} |\nabla u(t)|^2\, dt$$

$$\leqq \frac{1+C^{-1}}{\Gamma_n(1-\beta_2)} \int\limits_{|t|\geqq R_1} \left\{ r(t)^{1-\beta_2} \frac{V_2(t)}{a(t)} \right\} \sum_{i,j=1}^{n} a_{ij}(t)\; \partial_i \bar{u}(t)\; \partial_j u(t)\, dt$$

$$\leqq \mu \int\limits_{|t|\geqq R_1} \sum_{i,j=1}^{n} \partial_i \bar{u}(t)\; \partial_j u(t)\, dt\,.$$

Since $s(t)=\delta$ for $|t|\leqq R_1$, we get by (9), (iii), (v) and 2 (i)

$$(12)\quad \frac{(1+C^{-1})(1-\varepsilon)^{1-\beta_2}}{\Gamma_n(1-\beta_2)} \int\limits_{|t|\leqq R_1} \left\{ \int\limits_{\Sigma_{t,s_\varepsilon}} s(x)^{1-\beta_2}\, |v(x)|\; |x-t|^{1-n+\beta_2}\, dx \right\} |\nabla u(t)|^2\, dt$$

$$\leqq \frac{(1+C^{-1})\delta^{1-\beta_2}}{\Gamma_n(1-\beta_2)} \int\limits_{|t|\leqq R_1} \frac{V_2(t)}{a(t)} \sum_{i,j=1}^{n} a_{ij}(t)\; \partial_i \bar{u}(t)\; \partial_j u(t)\, dt$$

$$\leqq K\delta^{1-\beta_2} \int\limits_{|t|\leqq R_1} \sum_{i,j=1}^{n} a_{ij}(t)\; \partial_i \bar{u}(t)\; \partial_j u(rt)\, dt$$

$$\leqq \mu \int\limits_{|t|\leqq R_1} \sum_{i,j=1}^{n} a_{ij}(t)\; \partial_i \bar{u}(t)\; \partial_j u(t)\, dt\,,$$

where the last estimate is obtained by choosing δ sufficiently small, say $\delta=\delta_0$.

Finally, by (9) and (v) we get

$$(13)\quad \frac{(1+C)(1-\varepsilon)^{-1-\beta_1}}{\Gamma_n(1-\beta_1)} \int\limits_{|t|\leqq R_2} \left\{ \int\limits_{\Sigma_{t,s_\varepsilon}} s(x)^{-1-\beta_1}\, |v(x)|\; |x-t|^{1-n+\beta_1}\, dx \right\} |u(t)|^2\, dt$$

$$\leqq \frac{(1+C)(1-\varepsilon)^{-2-2\beta_1}\delta_0^{-1-\beta_1}}{\Gamma_n(1-\beta_1)} \int\limits_{|t|\leqq R_2} V_1(t)\; |u(t)|^2\, dt \;\leqq\; K\|u\|^2\,.$$

By addition of the inequalities (10)–(13) and insertion in (5) we arrive at

$$(14)\qquad \int\limits_{\mathbf{R}^n} |v|\; |u|^2 \;\leqq\; \mu \int\limits_{\mathbf{R}^n} \Big(\sum_{i,j=1}^{n} a_{ij}\; \partial_i \bar{u}\; \partial_j u + q\,|u|^2 \Big) \;+\; K\|u\|^2\,,$$

and the Lemma is proved.

It follows from Lemma 3 that the $\boldsymbol{M}_0$- and $\boldsymbol{L}_0$-metrics are equivalent on C_0^1 and $\boldsymbol{L}_0$ is bounded below. We denote by $\boldsymbol{L}$ the closure of $\boldsymbol{L}_0$ and by L the corresponding self-adjoint operator. It is clear that $D[\boldsymbol{L}]=D[\boldsymbol{M}]$, and by application of Lemma 1.2 and Lemma 2 we have proved the following.

THEOREM. *Under the assumptions* (i)–(ii) *of Section* 2 *and* (i)–(vi) *of Section* 3 *the operator* L *has discrete spectrum.*

REMARK 1. The Theorem holds true also if $\mathscr{L}$ is modified by addition of first order terms

$$\mathscr{L} = -\Delta + i\sum_{j=1}^{n} b_j \partial_j + q + v ,$$

where we have set $a_{ij}(x) = \delta_{ij}$ for simplicity; the functions q and v satisfy the previous conditions. The coefficients b_j are assumed to be real-valued functions in C^1 satisfying the conditions:

(i) $\sum_{j=1}^{n} \partial_j b_j = 0$.
(ii) There exists $a > 0$ such that

$$\sup_{x\in \mathsf{R}^n} \int_{S_{x,1}} |b_j(t)|^2 \, |t-x|^{2-n-a} \, dt < \infty, \qquad j = 1,\ldots,n .$$

This is proved by establishing the additional inequality

$$\int_{\mathsf{R}^n} \left| \sum_{j=1}^{n} b_j \, \partial_j u \right|^2 < \varepsilon \int_{\mathsf{R}^n} |\nabla u|^2 + K \|u\|^2$$

for $\varepsilon < 1$, by a proof similar to that of Lemma 3.

REMARK 2. The operator $\mathscr{L}$ is essentially self-adjoint on C_0^∞, if the coefficients satisfy certain weak conditions (cf. Stetkær-Hansen [5]). If $a_{ij}(x) = \delta_{ij}$, it suffices that for some $\alpha > 0$ the function

$$\int_{S_{x,1}} |q(t) + v(t)|^2 \, |t-x|^{4-n-\alpha} \, dt$$

be locally bounded.

EXAMPLE. Fix $\varepsilon > 0$ and set

$$\mathscr{L} = -\Delta + |x|^2 - \sum_{m=1}^{\infty} \frac{|x_m|^{1-\varepsilon}}{|x-x_m|} X_m(x) ,$$

where

$$|x_p - x_q| > 2 \quad \text{for } 1 \leqq p < q < \infty$$

and

$$X_m(x) = \begin{cases} 1 & \text{for } |x-x_m| \leqq 1 , \\ 0 & \text{for } |x-x_m| > 1 . \end{cases}$$

For $n \geqq 3$, the operator $\mathscr{L}$ is essentially self-adjoint on C_0^∞, and the corresponding operator L has discrete spectrum. Our Theorem applies with $q(x) = |x|^2 + 1$ and $r(x) = \min(|x|^{-1}, 1)$.

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