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MARTINGALE CONVERGENCE AND THE RADON-NIKODYM THEOREM IN BANACH SPACES

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1. Introduction.

In recent years, several authors have considered various extensions of the martingale convergence theorems of Doob [9] to the case where the random variables take values in a Banach space (B-space), e.g. Chatterji [4] [5], Scalora [18], A. I. and C. I. Tulcea [19], and Metivier [13]; the last named author has even considered the general case of locally convex topological vector spaces. Whereas certain types of convergence theorems were shown to be valid [4] [5] for arbitrary B-spaces, a counter-example in Chatterji [4] shows that without some condition on the B-space concerned, some of the most important convergence theorems of the scalar-valued case are invalid. The main purpose of this paper is to elucidate this latter situation, by demonstrating that the validity of almost any general theorem for martingales taking values in a B-space is equivalent to the fact that the Radon–Nikodym theorem is valid for set-functions taking values in such spaces. At the same time, this paper offers self-contained proofs of almost everywhere (a.e.) convergence theorems for B-space-valued martingales, theorems which are more general than those to be found in [18], [19]. The method of proof yields, as a by-product, several known Radon–Nikodym theorems for B-spaces, including one due to Phillips [14].

2. Notation and preliminary remarks.

For the sake of clarity of exposition, I shall consider only the case where the underlying measure space is a probability space S, with a σ-algebra Σ of measurable subsets and a σ-additive positive measure P on Σ with $P(S)=1$. Suitable generalizations to the case of an arbitrary

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measure space will be obvious to the interested reader. The letter X will be used to denote a B-space with norm $|\cdot|$ and all random variables f with values in X will be assumed to be strongly (or Bochner) measurable functions on S with values in X. The integral of such a function, denoted by $E(f)$ or $\int f(s)P(ds)$ or simply by $\int f$, will always be considered in the Bochner sense. These and other measure-theoretic concepts and notations are to be found in Dunford and Schwartz [10] or Hille and Phillips [12].

Given a sub-σ-algebra Σ_i of Σ, there exists a well-defined linear operator of norm one, the conditional expectation operator E_i, mapping $L^1(\Sigma,X) \to L^1(\Sigma_i,X)$ and satisfying

$$\int_A f = \int_A E_i f, \qquad A \in \Sigma_i .$$

Here $L^1(\Sigma,X)=\{f \mid f$ is Σ-measurable, $\|f\|_1=\int|f|<+\infty\}$. If $f=\sum_{k=1}^n a_k C_{A_k}(s)$, $a_k \in X$, $A_k \in \Sigma$ ($C_A(s)=1$ if $s \in A$ and 0 if $s \notin A$), then $E_i f=\sum_{k=1}^n P_i(A_k)a_k$ where P_i stands for conditional probability given Σ_i as in Doob [9]. By a standard approximation argument, for a general f, $E_i f$ can be easily shown to exist. This procedure is necessary since, given an X-valued σ-additive set-function μ on Σ such that $\mu(A)=0$ whenever $P(A)=0$, μ is not necessarily an indefinite integral of a function with respect to P, even though the total variation

$$V_\mu(A) = \sup\left\{\sum_{k=1}^n |\mu(A_k)| \mid A_k \in \Sigma,\ A_k \subset A,\ A_k \text{ disjoint}\right\},$$

which is always a non-negative measure on Σ, is totally-finite. Thus the standard argument for the existence of the conditional expectation operator E_i is not applicable. It is convenient to introduce at this point the following definition.

Definition 1. The B-space X has the RN-property with respect to (S,Σ,P) if every X-valued σ-additive set-function μ of bounded variation (that is, $V_\mu(S)<\infty$) which is absolutely continuous with respect to P (that is, $P(A)=0 \Rightarrow \mu(A)=0$ or equivalently, $V_\mu \ll P$) has an integral representation, that is,

$$\exists f \in L^1(\Sigma,X) \text{ such that } \mu(A) = \int_A f(s)\,P(ds) \ \ \forall A \in \Sigma .$$

The space X will be said to have property (D) if it has the RN-property with respect to Lebesgue measure on the Borel sets of the unit interval.

Bochner and Taylor [2] have defined property (D) for a B-space X as the property that a function of strong bounded variation on the unit interval is (strongly) differentiable almost everywhere. It can be easily seen from the methods of the present paper that their definition of property (D) is equivalent to mine.

It will follow from the considerations in the next section that, if P is not purely atomic, then X has the RN-property with respect to (S,Σ,P) if and only if X has property (D). So for all practical purposes, in this connection property (D) is what really matters. If P is purely atomic, then any B-space X has the RN-property with respect to (S,Σ,P), as can be immediately verified.

DEFINITION 2. Given a directed set $(N, \leqq)$ and a family of σ-algebras $\Sigma_i \subset \Sigma$, $i \in N$, the system $\{f_i, \Sigma_i, i \in N\}$ forms an X-valued martingale if $f_i \in L^1(\Sigma_i, X)$, $i \leqq j \Rightarrow \Sigma_i \subset \Sigma_j$, and $E_i f_j = f_i$.

The following two special examples of X-valued martingales will play special roles:

EXAMPLE (i). Let Σ_i, N be as above and let $f \in L^1(\Sigma, X)$. If $f_i = E_i f$ then $\{f_i, \Sigma_i, i \in N\}$ is an X-valued martingale.

EXAMPLE (ii). Let μ be an X-valued σ-additive set-function and let I be the directed set of all partitions $\pi = \{A_1, A_2, \ldots, A_n\}$ of S where $n \geqq 1$, $A_i \in \Sigma$, $P(A_i) > 0$, $\bigcup_{i=1}^n A_i = S$, A_i's disjoint. We write $\pi_1 \leqq \pi_2$ if every set in the partition π_2 is contained (P-almost surely) in a set of the partition π_1. Define

$$f_\pi(s) = \mu(A_i)/P(A_i) \qquad \text{if } s \in A_i .$$

Then $\{f_\pi, \Sigma_\pi, \pi \in I\}$ is an X-valued martingale where Σ_π is the σ-algebra generated by the sets of the partition π. For this to hold the additivity of μ is actually all that is necessary. These f_π martingales have often been used in measure theory; see e.g. Dunford and Schwarz [10, pp. 297].

As an illustration of the connection between the convergence of martingales and the RN-property, I shall state the following result which is of an elementary nature.

THEOREM 1. (a) *Let $f \in L^p(\Sigma, X)$, that is, f is Σ-measurable and $\|f\|_p^p = \int |f|^p < \infty$, $1 \leqq p < \infty$. Then for any directed set N and σ-algebras Σ_i, the martingale $\{f_i, \Sigma_i, i \in N\}$ of example* (i) *has the property*

$$\lim_i \|f_i - f_\infty\|_p = 0 ,$$

where $f_\infty = E_\infty f$ denotes the conditional expectation of f given the σ-algebra Σ_∞ generated by $\bigcup_{i \in N} \Sigma_i$.

(b) *In example* (ii), *if* $\mu(A) = \int_A f(s)\, P(ds)$, $f \in L^p(\Sigma, X)$, *then*

$$\lim_\pi \|f_\pi - f\|_p = 0 .$$

(c) *In example* (ii), *if* $\lim_{\pi_1, \pi_2} \|f_{\pi_1} - f_{\pi_2}\|_p = 0$, *that is, if* f_π *is an* L^p-*Cauchy sequence, then* $\mu(A) = \int_A f(s) P(ds)$ *for some* $f \in L^p(\Sigma, X)$.

REMARKS. Theorem 1(a) is a generalization to directed sets of a corresponding theorem in [5] where N is the set of all positive integers. Since the method of proof is exactly the same and in any case of utter simplicity, only a bare sketch will be provided. Parts (b) and (c) were proved slightly differently by Rønnow [17] for the case $p = 1$. Here (b) is an immediate corollary of (a) since $f_\pi = E_\pi f$ and clearly $\Sigma_\infty = \Sigma$ in this case. As regards (c), it will be noticed that when $1 < p < \infty$ and X is the set of complex numbers, the much weaker condition that $\sup_\pi \|f_\pi\|_p < +\infty$ is sufficient (and clearly always necessary) for the conclusion. This is indeed a classical theorem of F. Riesz where the condition is expressed as

$$\sup_\pi \sum_{i=1}^{n} \frac{|\mu(A_i)|^p}{[P(A_i)]^{p-1}} < \infty .$$

This latter assertion (not valid even in the classical case for $p = 1$) will follow from the main theorem of this paper for a wide class of spaces X; in fact, it shows, in a sense, exactly which spaces X allow such a theorem.

PROOF OF THEOREM 1. (a) Assume first that f is Σ_∞-measurable. If f is measurable with respect to the algebra $\bigcup_{i \in N} \Sigma_i$ then $E_i f = f$ for $i \geqq i_0$. Hence for this case the conclusion follows. A general f which is Σ_∞-measurable can be approximated arbitrarily closely in L^p-norm by functions measurable $\bigcup_{i \in N} \Sigma_i$. So the conclusion holds for such f. Finally, for any $f \in L^p(\Sigma, X)$,

$$f_i = E_i f = E_i E_\infty f = E_i f_\infty .$$

As pointed out above, (b) follows immediately.

(c) From the completeness of $L^p(\Sigma, X)$ it follows that $\exists f \in L^p(\Sigma, X)$ such that $\lim_\pi \|f_\pi - f\|_p = 0$. I shall now show that $f_\pi = E_\pi f$. Assertion (a) then will justify the conclusion of (c). Now given $\varepsilon > 0$, $\exists \pi_\varepsilon$ such that $\|f_{\pi'} - f\|_p < \varepsilon$ if $\pi' \geqq \pi_\varepsilon$. To any π, since the set I of partitions is directed, there is a partition π_1 which is finer than both π and π_ε, that is, $\pi_1 \geqq \pi$, $\pi_1 \geqq \pi_\varepsilon$. It has already been remarked that $\{f_{\pi'}, \Sigma_{\pi'}, \pi' \in I\}$ is a martingale and hence for any set $A \in \pi$

$$\int_A f_\pi = \int_A f_{\pi_1} .$$

Now

$$\left|\int_A f_\pi - \int_A f\right| = \left|\int_A f_{\pi_1} - \int_A f\right| \leqq \|f_{\pi_1} - f\|_1 \leqq \|f_{\pi_1} - f\|_p < \varepsilon .$$

Since ε is arbitrary and f_π is Σ_π-measurable, $E_\pi f = f_\pi$. This concludes the proof. An interesting corollary, noted by Rønnow [17] in the case $p=1$, will be stated here for later application.

Corollary. *In order that an additive X-valued set function μ be the integral of a function $f \in L^p(\Sigma, X)$, either one of the following two conditions is necessary and sufficient:*

(1) *For every monotone sequence π_n of partitions (that is, $\pi_n \leqq \pi_{n+1}$) the functions f_{π_n}, $n \geqq 1$, as defined in example* (ii) *above are Cauchy convergent in L^p.*

(2) *The restriction of μ to any separable σ-subalgebra of Σ (that is, one generated by a denumerable number of sets) has an integral representation by means of a function from $L^p(\Sigma, X)$.*

3. Discussion of the RN-property.

If P is purely atomic, that is, if there exists a sequence of disjoint sets $E_n \in \Sigma$, $P(E_n) > 0$, $P(\bigcup_{n=1}^\infty E_n) = 1$ such that the E_n's are P-atoms in Σ (that is, $F \in \Sigma$, $F \subset E_n$ implies $P(F) = 0$ or $P(E_n)$), then every B-space X has the RN-property with respect to (S, Σ, P). Indeed given any σ-additive, P-absolutely continuous, X-valued set function μ of bounded variation, the function

$$f(s) = \sum_{n=1}^\infty a_n C_{E_n}(s) \quad \text{with} \quad a_n = \mu(E_n)/P(E_n)$$

is easily seen to be an integrable function such that

$$\mu(E) = \int_E f \quad \text{for all } E \in \Sigma .$$

Now an arbitrary probability measure can be written down, essentially uniquely, as a convex combination

$$dP_1 + (1-d)P_2, \quad 0 \leqq d \leqq 1 ,$$

of two probability measures P_1, P_2 where P_1 is purely atomic and P_2 is purely nonatomic. It follows, therefore, that X will have the RN-property with respect to (S, Σ, P) if and only if it possesses the RN-property with respect to (S, Σ, P_2).

I shall assume now that P is purely nonatomic on Σ. By virtue of the corollary of the last section, X will possess the RN-property with respect to (S,Σ,P) if and only if this happens with respect to (S,Σ_0,P) for every separable σ-subalgebra Σ_0. Clearly, Σ_0 can be so chosen that, when restricted to Σ_0, P is also purely non-atomic. For instance, Σ_0 can be defined to be the σ-algebra generated by a sequence π_n of successively finer partitions such that $\pi_n = \{A_{n1}, A_{n2}, \ldots, A_{n2n}\}$ and $P(A_{nk}) = 2^{-n}$ for $n \geqq 1$. This is possible since P is nonatomic. Now if A is any set belonging to Σ_0 with $P(A) > 0$ then there exist indices n and k such that $0 < P(AA_{nk}) < P(A)$ which proves the non-existence of atoms in Σ_0. By a theorem of Halmos and von Neumann [11, p. 173] the measure algebra $(\tilde{\Sigma}_0, P)$ is isomorphic with the measure algebra $(\tilde{\mathscr{B}}, m)$ of the unit interval with Lebesgue measure m on the Borel sets. It is easy to see that the measure algebra isomorphism T between $\tilde{\Sigma}_0$ and $\tilde{\mathscr{B}}$ can be extended to an isometry between the whole of $L^1(\Sigma_0, X)$ and $L^1(\mathscr{B}, X)$ (considered as equivalence classes of functions) in such a way that $\int_A f dP = \int_{TA} Tf dm$ holds. It is to be noted that T is to be thought of as operating on equivalence classes of X-valued functions and that no assumption is made concerning the possibility of inducing the measure algebra isomorphism T through a 1-1 point transformation between S and the unit interval. Such a transformation—which may be impossible if S is "pathological"—is not necessary in the present discussion. Since any X-valued σ-additive P-absolutely continuous (m-absolutely continuous) set function μ can be lifted to the respective measure algebras $\tilde{\Sigma}_0(\tilde{\mathscr{B}})$, it is clear from the above that X has property (D) if and only if X has the RN-property with respect to (S, Σ_0, P).

I shall now summarize the conclusions of the above discussion in the form of a theorem:

THEOREM 2. (a) *If (S,Σ,P) is purely atomic then every B-space has the* RN-*property with respect to it.*

(b) *If P is not purely atomic then a B-space has the property* (D) *if and only if it has the* RN-*property with respect to (S,Σ,P).*

Thus we see that the RN-property is really independent of the underlying probability space and can be considered entirely in relation to the unit interval.

4. Preliminary a.e. convergence theorems.

The purpose of this section is to prove a convergence theorem which ensures a.e. convergence of the martingales of Theorem 2(a) above in the case of the directed set $\mathsf{N}=\{1,2,3,\ldots\}$ under the natural ordering. No assumptions about the space X are necessary for this theorem. In this generality, the theorem was first proved, using a deep theorem of Banach, in Chatterji [5] and later also by. A. I. and C. I. Tulcea [19]. The proof presented here is totally elementary and depends on the following lemma which is stated in the present form for later use.

LEMMA 1. *Let $\{f_n,\Sigma_n,\ n\geqq 1\}$ be an X-valued martingale and let $A\in\Sigma_N$. Then for any $\varepsilon>0$,*

$$P\{s\in A,\ \sup_{n\geqq N}|f_n(s)|\geqq\varepsilon\}\ \leqq\ \varepsilon^{-1}\sup_{n\geqq N}\textstyle\int_A|f_n|\ .$$

The lemma is an easy consequence of the fact that $|f_n|$ is a positive submartingale and is, in this sense, well known; see Doob [9, p. 314].

THEOREM 3. *Let $f\in L^1(\Sigma,X)$ and let $f_n=E_nf$ denote conditional expectation with respect to Σ_n. Let $\Sigma_n\subset\Sigma_{n+1}$, $n=1,2,\ldots$. Then*

$$\lim_{n\to\infty}f_n\ =\ f_\infty$$

exists (strongly) a.e. and $f_\infty=E_\infty f$ equals the conditional expectation of f given Σ_∞, the σ-algebra generated by the algebra $\bigcup_1^\infty\Sigma_n$.

PROOF. Since the proof is exactly the same as one of the proofs for the scalar-valued case (see Billingsley [1] or Dunford and Schwartz [10, p. 208]) it will be presented only briefly here. If f is measurable with respect to $\bigcup_1^\infty\Sigma_n$ then $f_n=f$ from some point on and hence the conclusion above is immediate. If f is measurable Σ_∞ then, given $\varepsilon>0$, $\delta>0$, a function g can be found, measurable $\bigcup_1^\infty\Sigma_n$, and such that $\|f-g\|_1<\frac{1}{2}\varepsilon\delta$. By the linearity of the operators E_n, one has

$$\begin{aligned}|f_n-f_m|\ &\leqq\ |E_ng-E_mg|+|E_n(f-g)-E_m(f-g|\\ &\leqq\ |E_ng-E_mg|+2\sup_{n\geqq1}E_n|f-g|\ .\end{aligned}$$

Hence

$$\limsup_{m,n\to\infty}|f_n-f_m|\ \leqq\ h\ =\ 2\sup_{n\geqq1}E_n|f-g|$$

so that

$$P\{\limsup_{m,n\to\infty}|f_n-f_m|\geqq\varepsilon\}\ \leqq\ P\{h\geqq\varepsilon\}\ \leqq\ 2\varepsilon^{-1}\|f-g\|\ <\ \delta$$

by an application of Lemma 1 to the real-valued martingale $E_n|f-g|$. Since δ is arbitrary,

$$P\{\limsup_{m,n\to\infty}|f_n-f_m|\geqq\varepsilon\}\ =\ 0$$

whence, ε being arbitrary, the existence of $\lim_{n\to\infty}f_n$ is demonstrated. For a general $f\in L^1(\Sigma,X)$, since $f_n=E_nf=E_nf_\infty$ and f_∞ is Σ_∞-measurable, the existence of $\lim f_n$ is assured. The identification of the limit as f_∞ follows immediately from Theorem 2(a) above.

For general reference, I shall state a theorem here for the case $N = \{0, -1, -2, \ldots\}$ which was proved in [5], again by the afore-mentioned theorem of Banach, and can now be proved by the method indicated above, without any use of scalar-valued martingale theory.

THEOREM 4. *Let $\{f_n, \Sigma_n, n \leqq 0\}$ be an X-valued martingale; then*

$$\lim_{n\to-\infty} f_n = f_{-\infty}$$

exists strongly a.e. *and also in $L^1(\Sigma, X)$, where $f_{-\infty} = E_{-\infty} f_0$ denotes the conditional expectation of f_0 given $\Sigma_{-\infty} = \bigcap_{n\leqq 0} \Sigma_n$.*

It may be appropriate to add here that generalizations of Theorems 3 and 4 to arbitrary index sets N are not possible, even in the scalar-valued case, without some further assumptions on the structure of the σ-algebras Σ_n. The first counter-example was given by Dieudonné [8]. A much simpler counter-example has recently been given by Chow [6]. I should like to point out here a more obvious way of looking at Chow's example. Let $\{g_n, n \geqq 1\}$ be a sequence of independent r.v.'s with $E(g_n) = 0$, taking values in an arbitrary B-space X. Let $f = \sum g_n$ exist a.e. but suppose that the series is almost surely not unconditionally convergent. Let further $f \in L^1(\Sigma, X)$. Define $f_\pi = \sum_{n\in\pi} g_n$ where π is a finite set of positive integers. Let the π's be ordered by inclusion. If Σ_π is the smallest σ-subalgebra with respect to which $\{g_n,\ n \in \pi\}$ are measurable, then clearly $f_\pi = E_\pi f$. Further, $\lim_\pi f_\pi$ cannot exist almost surely since this is equivalent to the unconditional convergence of $\sum g_n$ almost surely. Note, however, that Theorem 2(a) implies that $\|f_\pi - f\|_1 \to 0$ all the same. A convenient way of choosing g_n is to take $g_n = \varepsilon_n a/n$ where $0 \neq a \in X$ and the random variables ε_n equal ± 1 with probability $\frac{1}{2}$ and are independent. In this case, $f \in L^2(\Sigma, X)$ since $E|f|^2 = |a| \sum 1/n^2 < \infty$ and hence, by Theorem 2(a), f_π converges to f in $L^2(\Sigma, X)$. This choice was made by Chow [5, p. 1490], but the point made here is that no calculation is necessary to show that $\lim f_\pi$ does not exist since the series $\sum g_n$ is blatantly not unconditionally convergent. In the real-valued case this latter fact automatically implies that $\limsup f_\pi = +\infty$ and $\liminf f_\pi = -\infty$. A counter-example to Theorem 4, that is, the decreasing index case, is also possible. Consider "Riemann sums"

$$f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(x + k/n), \quad \text{where} \quad f \in L^1(0,1)$$

with respect to Lebesgue measure and + denotes addition modulo 1. Then it is easy to verify that $f_n = E_n f$ is the conditional expectation of f given Σ_n, the σ-algebra of Borel-sets of the unit interval with period $1/n$. If $n_1 | n_2$ then $\Sigma_{n_1} \supset \Sigma_{n_2}$. Define $n_2 \ll n_1$ if $n_1 | n_2$. Then $\{f_n, \Sigma_n\}$ is a martingale which need not converge a.e., as shown by the counter-example in Rudin [16], even though $f \in L^\infty(0,1)$. The analogue of Theorem 2(a), however, shows that in all cases

$$f_n \to a \text{ in } L^1(0,1), \quad \text{where} \quad a = \int_0^1 f.$$

5. A decomposition theorem for X-valued set functions.

In order to avoid interrupting the proof of the main theorem in the next section, I shall present here a theorem concerning finitely additive X-valued set functions. As proto-type for this theorem, in the scalar-valued case, can be considered a theorem of Hewitt and Yosida which states that every finitely additive (scalar) set function on an algebra can be uniquely decomposed into the sum of a σ-additive and a purely finitely additive set function. A convenient reference is [10, pp. 163–64]. The present theorem for X-valued set functions is not as sharp as the above theorem but is enough for my purposes.

THEOREM 5. *Let P be a probability measure on (S,Σ) where Σ is only assumed to be an algebra of sets, and let μ be an X-valued finitely additive set function of Σ, of bounded total variation. Then $\mu = \sigma + \eta$ where σ is a σ-additive set function whose total variation V_σ is finite and P-absolutely continuous while η is a finitely additive set function whose total variation V_η is finite and P-singular; that is, given $\varepsilon, \delta > 0$ there exists an $A \in \Sigma$ such that*

$$P(A) < \varepsilon \quad \textit{and} \quad V_\eta(A') < \delta ,$$

A' denoting the complement of A.

PROOF. The method to be used is fairly standard and is incorporated in pp. 311–13 of Dunford and Schwartz [10]. Given the space (S,Σ), there is a space S_1 which is a compact Hausdorff space with the following properties:

(1) S_1 is totally disconnected, that is, the algebra Σ_1 of simultaneously closed and open (clopen) sets form a basis for the topology of S_1;

(2) there is an isometric isomorphism H between $B(S,\Sigma)$, the space of bounded scalar-valued Σ-measurable functions on S, and $C(S_1)$, the space

of scalar-valued continuous functions on S_1, both spaces being considered under the uniform norm.

Let the correspondence $H(C_A(s)) = C_{A_1}(s_1)$ (the C's standing for characteristic functions) induce the set algebra isomorphism τ between Σ and Σ_1, that is, define $\tau(A) = A_1$. This correspondence is such that $\tau(\Sigma) = \Sigma_1$. Now given an additive or σ-additive (X-valued or scalar-valued) set function Q on Σ, the formula

$$Q_1(A_1) = Q(\tau^{-1}(A_1))$$

always defines a σ-additive set function on Σ_1, whether or not Q was σ-additive to start with. The reason for this is that the σ-additivity equation for Q_1, viz.,

$$Q_1\left(\bigcup_1^\infty A_n\right) = \sum_1^\infty Q_1(A_n)$$

if $A_n \in \Sigma_1$, A_n's disjoint and $\bigcup_1^\infty A_n \in \Sigma_1$, is trivially satisfied since the compactness of S_1 precludes the existence of an infinite sequence of non-empty disjoint A_n's $\in \Sigma_1$ such that $\bigcup_1^\infty A_n \in \Sigma_1$ also. Clearly if Q is of finite total variation, so is Q_1 on Σ_1. If this is so, then Q_1 can be extended to the σ-algebra Σ_2 generated by Σ_1. If Q is scalar-valued, this is possible by a classical theorem of Caratheodory. If Q_1 is X-valued then also the fact mentioned has been known for a long time; for convenient reference, see [19, p. 119, foot-note (6)]. Now let P_1, μ_1 be the transpositions of the P, μ of the theorem to the space (S_1, Σ_1). Let P_1, μ_1 stand also for the extended set functions on (S_1, Σ_2). The set function μ_1 is then of bounded total variation on Σ_2.

According to a theorem of Rickart [15] which generalizes the classical Lebesgue decomposition theorem for scalar-valued set functions, μ_1 is of form $\sigma_1 + \eta_1$ on the σ-algebra Σ_2 where σ_1, η_1 are of bounded variation if μ_1 is so (as in this case), σ_1 is P_1-absolutely continuous, and η_1 is P_1-singular. Let σ, η be the inverse images of the restrictions of σ_1, η_1 to Σ_1. Then on the given space (S, Σ), $\mu = \sigma + \eta$ where V_σ is P-absolutely continuous and V_η is P-singular. The σ-additivity of σ follows trivially from the fact that V_σ is absolutely continuous with respect to a σ-additive function P. Thus the decomposition theorem is completely established.

It seems likely that η should be further decomposable into a sum of two set functions, one σ-additive and P-singular and the other purely finitely additive by which is meant that its total variation is singular with respect to all σ-additive set functions on Σ. I have not been able to prove this yet.

6. Main theorem.

THEOREM 6. *For a B-space X and a probability space (S,Σ,P) the following statements are equivalent when holding for all X-valued martingales $\{f_n,\Sigma_n\}$, $n \geqq 1$:*

(1) *If* $\sup_{n\geqq 1}\|f_n\|_1 < +\infty$, *then* $f_\infty = \lim_{n\to\infty} f_n$ *exists strongly a.e.*

(2) *If* $\sup_{n\geqq 1}\|f_n\|_1 < +\infty$, *then* $f_\infty = \lim_{n\to\infty} f_n$ *exists weakly a.e., in the sense that* $\exists f_\infty$ *strongly measurable such that for all* $y^* \in X^*$,

$$\lim_{n\to\infty}\langle f_n(s),y^*\rangle = \langle f_\infty(s),y^*\rangle \quad \textit{for } s \notin N_{y^*}, P(N_{y^*})=0 .$$

It is enough to know that f_∞ is a.e. separable-valued to deduce a version of it which is strongly measurable. See proof of the subsequent Theorem 7 for an elucidation of this condition.

(3) *If* $\sup_{n\geqq 1}|f_n(s)| < C$ *a.e. for some* $C>0$, *then* $f_\infty = \lim_{n\to\infty} f_n$ *exists strongly a.e.*

(4) *If for some* $C>0$, $\sup_{n\geqq 1}|f_n(s)| < C$ *a.e., then* $f_\infty = \lim_{n\to\infty} f_n$ *exists weakly a.e. in the sense of statement* (2).

(5) *If the* f_n*'s are uniformly integrable (that is,* $\lim_{N\to\infty}\int |f_n| C_{\{|f_n|>N\}} = 0$ *uniformly in* $n \geqq 1$), *then* $\exists f_\infty \in L^1(\Sigma,X)$ *with* $\lim_{n\to\infty}\|f_n - f_\infty\|_1 = 0$.

(6) *If* $\sup_{n\geqq 1}\|f_n\|_p < \infty$, $1<p<\infty$, *then* $\exists f_\infty \in L^p(\Sigma,X)$ *with* $\lim_{n\to\infty}\|f_n - f_\infty\|_p = 0$.

(7) *The space X has the* RN-*property with respect to* (S,Σ,P).

REMARK. The reader is reminded that in view of the discussion of the RN-property given above, the convergence properties of X-valued martingales are rather independent of the underlying probability space. If P is purely atomic, then all the 7 statements above hold for all B-spaces X. If P is not purely atomic and if X has one of the above 7 properties then X has all of them with respect to any other probability space and in particular X has property (D). I should like to remark that the equivalence of (5) and (7) have also been pointed out by Rønnow [17]. Some of the equivalences above (e.g. (2) $\Leftrightarrow$ (5)) can be deduced very easily, independently of the rest, and are listed for their possible utility and for completeness.

PROOF OF THEOREM 6. The major part of the proof consists in showing that (7) $\Rightarrow$ (1). All the other implications then follow by fairly routine arguments. So I begin with

PROOF OF (7) $\Rightarrow$ (1). Given the martingale $\{f_n,\Sigma_n, n \geqq 1\}$ with the property that

$$\sup_{n\geqq 1} E|f_n| < \infty ,$$

let the X-valued set function μ_n be defined on Σ_n by the formula

$$\mu_n(A) = \int_A f_n(s)\, P(ds)\,.$$

Clearly, the martingale property of the f_n's is equivalent to the statement that μ_{n+1} is an extension of μ_n to $\Sigma_{n+1} \supset \Sigma_n$. Hence the formula

$$\mu(A) = \lim \mu_n(A)$$

defines an X-valued set function on the algebra $\Sigma_\omega = \bigcup_1^\infty \Sigma_n$ which is clearly finitely additive. Let

$$V_\mu(A) = \sup\left\{\sum_{i=1}^{k} |\mu(B_i)| \mid B_i \in \Sigma_\omega,\ B_i \subset A,\ B_i \text{ disjoint},\ 1 \leqq k < \infty\right\}$$

be the total variation of μ for a set $A \in \Sigma_\omega$. It is easy to see that

$$V_\mu(A) = \lim \int_A |f_n| < +\infty\,.$$

In other words, μ is a finitely additive set function of bounded total variation on the algebra Σ_ω. One of the difficulties in proving (1) is that μ may not be σ-additive, a difficulty which may arise even in the scalar-valued case. I shall obviate this difficulty by using Theorem 5 of the preceeding section. According to that theorem $\mu = \sigma + \eta$ where σ is σ-additive and its variation is P-absolutely continuous. By the RN-property (that is, (7)),

$$\sigma(A) = \int_A g, \quad A \in \Sigma_\omega, \quad g \in L^1(\Sigma_\infty, X)\,,$$

Σ_∞ being the σ-algebra generated by Σ_ω. If σ_n is the restriction of σ to Σ_n then clearly

$$\sigma_n(A) = \int_A g_n, \quad A \in \Sigma_n\,,$$

where $g_n = E_n g$. Since, by assumption, the restriction μ_n of μ to Σ_n is also an integral, the restriction η_n of η to Σ_n must be of the form $\int_A h_n$. Indeed $f_n = g_n + h_n$, and $\{g_n, \Sigma_n\}$, $\{h_n, \Sigma_n\}$ are X-valued martingales. Moreover, since $g_n = E_n g$, by Theorem 3, $\lim g_n = g$ exists strongly a.e. I shall now show that $\lim h_n = 0$ strongly a.e. Because of the P-singularity of V_η, given $0 < \varepsilon$, $\delta < 1$, I can find $A \in \Sigma_\omega$ (and hence $A \in \Sigma_N$ for some N) such that

$$P(A') + V_\eta(A) < \tfrac{1}{2}\varepsilon\delta\,.$$

Now

$$\begin{aligned} P\{\sup_{n\geqq N}|h_n|>\varepsilon\} &= P\{A';\sup_{n\geqq N}|h_n|>\varepsilon\} + P\{A;\sup_{n\geqq N}|h_n|>\varepsilon\} \\ &< \tfrac{1}{2}\varepsilon\delta + \varepsilon^{-1}\sup_{n\geqq N}\int_A |h_n|\,P(ds) \quad \text{(by Lemma 1)} \\ &= \tfrac{1}{2}\varepsilon\delta + \varepsilon^{-1}V_\eta(A) < \tfrac{1}{2}\varepsilon\delta + \tfrac{1}{2}\delta < \delta\,. \end{aligned}$$

Hence

$$P\{\limsup_{n\to\infty}|h_n|>\varepsilon\} \leqq P\{\sup_{n\geqq N}|h_n|>\varepsilon\} < \delta\,.$$

The numbers ε,δ being arbitrary, it follows that $\lim|h_n|=0$ a.e. This proves that $\lim f_n$ exists strongly a.e. and to some extent characterizes the limit function.

Proof of (1) $\Rightarrow$ (5). Suppose the f_n's are uniformly integrable. Then $\sup_{n\geq 1}\|f_n\|_1<\infty$ and hence by (1) the limit $\lim f_n=f_\infty$ exists strongly a.e. Clearly $f_\infty \in L^1(\Sigma_\infty,X)$ since by Fatou's lemma

$$E|f_\infty| \leqq \lim\|f_n\|_1\,.$$

Hence $|f_n(s)-f_\infty(s)|$ as a sequence of real-valued functions is uniformly integrable and tends to 0 a.e. Therefore

$$\lim_{n\to\infty}\|f_n-f_\infty\|_1 = \lim_{n\to\infty}E|f_n-f_\infty| = 0\,.$$

Proof of (5) $\Rightarrow$ (7). By the Corollary to Theorem 1, given a P-absolutely continuous X-valued σ-additive function μ of bounded total variation on Σ, to prove that μ is a P-integral, it is enough to verify that for every sequence π_n of finer and finer partitions, the sequence of X-valued r.v.'s f_n (denoted there by f_{π_n}) which forms a martingale $\{f_n,\Sigma_n\}$ ($\Sigma_n=\sigma$-algebra formed by π_n), is such that the f_n's converge in $L^1(\Sigma,X)$. If I can show that the f_n's are uniformly integrable, then by virtue of (5), this latter will follow and (7) will be deduced. Because of the inequality

$$P(|f_n|\geqq N) \leqq N^{-1}\|f_n\|_1 \leqq N^{-1}V_\mu(S)\,,$$

given $\varepsilon>0$, one can choose N so large that

$$P(|f_n|\geqq N) < \varepsilon \quad \text{for all } n\geqq 1\,.$$

Because V_μ is P-absolutely continuous, given $\delta>0$, there exists an $\varepsilon>0$ such that

$$P(A)<\varepsilon \quad \text{implies} \quad V_\mu(A)<\delta \text{ for } A\in\Sigma\,.$$

Hence for any $\delta>0$,

$$\int_{\{|f_n|\geqq N\}} |f_n| \leqq V_\mu\{|f_n|\geqq N\} < \delta, \qquad n=1,2,\ldots,$$

if first ε and then N are chosen as indicated above.

This proves the uniform integrability of f_n, and hence (7).

Proof of (2) $\Rightarrow$ (5). Let $\{f_n,\Sigma_n\}$ be a uniformly integrable X-valued martingale. Clearly $\sup_{n\geqq 1}\|f_n\|_1<\infty$; hence by (2) there exists f_∞, which can be easily seen to be in $L^1(\Sigma_\infty,X)$, such that

$$\lim\langle f_n(s),y^*\rangle = \langle f_\infty(s),y^*\rangle \quad \text{a.e.}$$

for any $y^*\in X^*$. Since the uniform integrability of the f_n's clearly implies the same for $\langle f_n(s),y^*\rangle$, it follows that for every $y^*\in X^*$,

$$\{\langle f_n,y^*\rangle,\Sigma_n, 1\leqq n\leqq\infty\}$$

is a scalar-valued martingale and hence, in particular, for $A\in\Sigma_n$, the relation

$$\left\langle \int_A f_n, y^*\right\rangle = \int_A \langle f_n,y^*\rangle = \int_A \langle f_\infty,y^*\rangle = \left\langle \int_A f_\infty, y^*\right\rangle$$

is valid for every $y^*\in X^*$. Hence

$$\int_A f_n = \int_A f_\infty \quad \text{for all } A\in\Sigma_n .$$

In other words, $f_n=E_nf_\infty$. Theorem 1 then implies that $\|f_n-f_\infty\|_1\to 0$.

The implication (1) $\Rightarrow$ (2) being trivial, the above arguments show that (1), (2), (5), and (7) are equivalent.

Proof of (3) $\Rightarrow$ (7). If condition (3) holds for some $C>0$ then clearly it holds for all $0<C<\infty$. Suppose first that the X-valued set function μ is such that

$$\left\|\frac{dV_\mu}{dP}\right\|_\infty \leqq N, \quad \text{that is,} \quad P\left\{\frac{dV_\mu}{dP}\leqq N\right\} = 1$$

which means that $V_\mu(A)\leqq NP(A)$ for all $A\in\Sigma$ and some integer $N\geqq 1$. Because of the corollary to Theorem 1, as in the proof of (5) $\Rightarrow$ (7), it suffices to prove that for every sequence of increasingly finer partitions π_n, the associated martingale $\{f_n,\Sigma_n\}$ is such that the f_n's converge in $L^1(\Sigma,X)$. Since $V_\mu(A)\leqq NP(A)$, it follows that $\sup_{n\geqq 1}|f_n(s)|\leqq N$, and by (3) the f_n's converge strongly a.e. to a function f_∞ which is then automatically in $L^1(\Sigma,X)$. By the dominated convergence theorem, since $|f_n(s)-f_\infty(s)|\leqq 2N$ a.e.,

$$\|f_n - f_\infty\|_1 \to 0 .$$

Thus every X-valued set function μ under consideration, with the above-mentioned extra property, is representable as an integral. For a general μ, the proof now proceeds by a standard argument, which has nothing to do with martingale theory, as follows. Let

$$A_N = \left\{ s \,\middle|\, \frac{dV_\mu}{dP} \leqq N \right\}.$$

Clearly $A_N \subset A_{N+1}$ and $\Omega = \bigcup_1^\infty A_N$. Let $\mu_N(B) = \mu(BA_N)$ for $B \in \Sigma$. Then $V_{\mu_N}(B) = V_\mu(BA_N)$ and so

$$V_{\mu_N}(B) \leqq NP(BA_N) \leqq NP(B) .$$

By what has already been proved, it follows that

$$\mu_N(B) = \int_B f_N \quad \text{for some} \quad f_N \in L^1(\Sigma, X) .$$

It is easily seen that $f_N = 0$ a.e. on A_N' and that for $N > M$ $(A_N \supset A_M)$

$$f_N = f_M \text{ a.e.} \quad \text{on } A_M .$$

Hence for $N > M$,

$$\int |f_N - f_M| = \int_{A_M'} |f_N| = V_{\mu_N}(A_M') = V_\mu(A_N A_M') \leqq V_\mu(A_M')$$

so that

$$\|f_N - f_M\|_1 \to 0 \quad \text{as } M, N \to \infty .$$

Hence there exists an $f \in L^1(\Sigma, X)$ such that

$$\|f_N - f\|_1 \to 0 \quad \text{as } N \to \infty .$$

Since

$$\mu(B) = \lim_{N\to\infty} \mu(BA_N) = \lim_{N\to\infty} \int_B f_N = \int_B f(s) P(ds) ,$$

(7) is proved.

The argument of (2) $\Rightarrow$ (5) shows that (4) $\Rightarrow$ (3) since the condition in (3) implies uniform integrability and once it has been shown that there exists f_∞ such that $\|f_n - f_\infty\|_1 \to 0$, it follows that $f_n = E_n f_\infty$ whence Theorem 3 leads to the conclusion of (3).

Since the implications (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) are immediate, it follows that (1), (2), (3), (4), (5), and (7) are equivalent.

As regards (6), notice first that (6) $\Rightarrow$ (3) by an argument used already. For if $\sup_n |f_n(s)| < C$ a.e., then $\|f_n\|_p < C$ for $n \geqq 1$. Therefore by (6) there exists $f_\infty \in L^p(\Sigma, X)$ such that

$$\|f_n - f_\infty\|_p \to 0, \quad 1 < p < \infty .$$

It then follows that $f_n = E_n f_\infty$ and Theorem 3 does the rest.

On the other hand (5) $\Rightarrow$ (6) because, given a martingale $\{f_n, \Sigma_n\}$ with $\sup_{n \geqq 1} \|f_n\|_p < \infty$, $1 < p < \infty$, it follows immediately that the f_n's are uniformly integrable and hence by (5), there exists f_∞ such that $\|f_n - f_\infty\|_1 \to 0$. This implies as before that $f_n = E_n f_\infty$. Further $f_\infty \in L^p(\Sigma_\infty, X)$ since by Fatou's lemma

$$\int |f_\infty|^p \leqq \lim_{n \to \infty} \int |f_n|^p < \infty$$

by the assumption of (6). Theorem 1 now implies that $\|f_n - f_\infty\|_p \to 0$.

Thus the equivalence of (1)–(7) is established.

7. Applications.

In this section the main theorem will be used to deduce some well known Radon–Nikodym propositions for X-valued set functions. To emphasize the simplicity of these deductions, I should like to point out that what is needed is not the whole strength of the main theorem but rather the following elementary version of it. Let μ be an X-valued σ-additive set function of bounded total variation on the probability space (S, Σ, P) and let $\mu(A) = 0$ whenever $P(A) = 0$. Then for any sequence of partitions π_n, $n \geqq 1$, which become increasingly finer, the functions $f_{\pi_n}(s)$ of Example (ii) of section (2) are uniformly integrable. The set function μ has the integral representation $\int_A f(s) P(ds)$ if and only if for every sequence π_n of increasingly finer partitions the corresponding sequence f_{π_n} converges weakly a.e. (P) to a strongly measurable function $f_\infty(s)$ in the sense that for all $y^* \in X^*$, there is a set of P-measure zero $N_y{}^*$, possibly depending on y^*, such that if $s \notin N_y{}^*$ then $\lim \langle f_n(s), y^* \rangle = \langle f_\infty(s), y^* \rangle$. It is left to the interested reader to verify that the "non-elementary" argument (7) $\Rightarrow$ (1) of the main theorem is nowhere needed in a proof of the above statement.

Using this statement, I shall now derive a theorem originally due to Phillips [14]. A variety of other theorems of this sort, e.g. the Dunford–Pettis theorem and the Dunford–Pettis–Phillips theorem (see Bourbaki [3, p. 000]) follows effortlessly in a similar manner, without any separability assumption on the space X as was originally made, and later removed by the use of "lifting" arguments, by A. I. and C. I. Tulcea [20]. These and some more recent theorems of Mr. M. A. Rieffel (to be published) and representations by means of integrals other than Bochner-integrals will be deferred to a more systematic treatment in a later publication.

THEOREM 7 (see Phillips [14]). *Let μ be an X-valued σ-additive set function of totally bounded variation on a probability space (S,Σ,P) such that $\mu(A)=0$ whenever $P(A)=0$. If for every integer $N \geqq 1$ the set*

$$K_N = \left\{ \frac{\mu(A)}{P(A)} \,\middle|\, \frac{|\mu(A)|}{P(A)} \leqq N,\ P(A)>0 \right\}$$

is relatively weakly compact, then

$$\mu(A) = \int_A f(s)P(ds), \qquad \text{where } f \in L^1(\Sigma,P) .$$

PROOF. I shall suppose first that for some integer $N \geqq 1$, $|\mu(A)| \leqq NP(A)$ for all $A \in \Sigma$. The general proof can be derived from this special case exactly by means of the method sketched in the proof, (3) $\Rightarrow$ (7), of the main theorem. By virtue of the remarks made at the beginning of this section, it suffices to show that if π_n is an increasingly finer sequence of partitions of S, then the corresponding functions f_n converge weakly a.e. to a strongly measurable function f_∞ in the sense described before. Actually, it is enough to know that f_∞ is separable-valued a.e. to deduce its strong measurability since the limit relation

$$\lim_{n\to\infty} \langle f_n(s), y^* \rangle = \langle f_\infty(s), y^* \rangle \text{ a.e.}$$

(even if the null set depends on $y^* \in X^*$), implies that for each $y^* \in X^*$ the function $\langle f_\infty(s), y^* \rangle$ is measurable with respect to the σ-algebra Σ^*, the completion of Σ under the probability measure P. By a known theorem, (see Hille–Phillips [12]), f_∞ is then strongly measurable with respect to Σ^*. Clearly f_∞ can then be changed on a set of P-measure zero, so that the new version is Σ-strongly measurable and such that the weak convergence of f_n to f_∞ in the above sense remains unaltered.

From the definition of the f_n's it is to be seen that these finitely-valued r.v.'s take their values in the set defined in the statement of the theorem. Let X_0 be the closed separable linear manifold spanned by the values of $f_n(s)$, $s \in S$, $n \geqq 1$. Two things about X_0 are to be noticed.

(i) X_0 is automatically weakly closed, by a general theorem (see [10], p. 422, Theorem 13);

(ii) because of the hypothesis of Theorem 7, the subset of X_0 consisting of the values of $f_n(s)$ is relatively weakly compact.

For any point $s \in S$, let a subsequence n_k be chosen so that $f_{n_k}(s)$ converges weakly to $f_\infty(s)$, an element of X_0. This is possible because of (i) and (ii) above. (An application of the axiom of choice is involved in this procedure.) Now for any $y^* \in X^*$ the sequence $\langle f_n(s), y^* \rangle$, being a scalar-valued martingale, converges a.e. Hence

$$\lim_{n\to\infty}\langle f_n(s),y^*\rangle = \langle f_\infty(s),y^*\rangle \text{ a.e.}$$

Since $f_\infty(s)$ is separable-valued, the remarks made before show that it may be chosen to be strongly Σ-measurable. Hence the criterion given at the beginning of this section ensure that μ has an integral representation by means of a function from $L^1(\Sigma,X)$.

COROLLARY. *The following classes of B-spaces X have property* (D) *and hence the* RN*-property with respect to any probability space* (S,Σ,P):

(i) *the reflexive spaces,*

(ii) *separable duals X of Banach spaces* (*that is, X is separable and there is a B-space Y such that* $Y^*=X$),

(iii) *weakly complete spaces X with separable duals* (*that is, X is weakly complete and* X^* *is separable*).

PROOF. That the reflexive spaces have the property (D) follows immediately from Theorem 7. For the other two classes, the property (D) can be derived similarly. The details are omitted.

From the counter-example of the next section, it will be seen that neither separability nor weak completeness can be left out in the description of the classes (ii) and (iii). The classes (i)–(iii) have for some time been known to possess property (D). I hope to discuss property (D) in greater detail in a later publication.

8. A counter-example.

Several examples are known of X-valued set functions which are σ-additive, P-absolutely continuous, of totally bounded variation, but not integrals. For example, if S is the unit interval (with $P=$Lebesgue measure on $\Sigma=$Borel sets) and $X=L^1$ over this space, then $\mu(A)=C_A(x)\in L^1$ is an old instance. In Chatterji [4], a corresponding martingale is constructed in the obvious way; it converges almost nowhere in any sense. As [19] points out, this shows in particular that L^1 is not the dual of any space, by virtue of (ii) of the Corollary above, a fact first pointed out by Dieudonné. An example of a non-convergent martingale has been recently given by Rønnow [17]. I should like to present it here in a different and very simple form and in a way which illustrates various new features of the theory of X-valued r.v.'s. The underlying probability space is again that of the unit interval and the B-space c_0 involved is the space of real or complex sequences which converge to zero with

$$|x| = \sup_{j\geqq 1}|x_j|, \quad x = (x_1,x_2,\ldots)\,.$$

Let $\gamma_n(s)$ be the sequence of Rademacher functions on the unit interval, defined in the following way: let $s = \Sigma_1^\infty a_n(s)2^{-n}$ be the binary expansion of $0 \leqq s \leqq 1$; then

$$\gamma_n(s) = 1 - 2a_n(s) = \pm 1$$

with probabilities $\frac{1}{2}$. The Rademacher functions are known to be stochastically independent under Lebesgue measure. Let

$$e_n = (0,0,\ldots,1,0,\ldots) \in c_0 \text{ (1 at the } n\text{th place)}; \; |e_n| = 1, \quad n \geqq 1\,.$$

Define

$$f_n(s) = \sum \gamma_k(s)e_k = (\gamma_1(s),\gamma_2(s),\ldots,\gamma_n(s),0,\ldots)\,.$$

It is immediate that $\{f_n,\Sigma_n\}_{n\geqq 1}$ is a martingale, where Σ_n is the σ-algebra generated by intervals of the type $(k2^{-n},(k+1)2^{-n})$, $0 \leqq k \leqq 2^n - 1$. Actually f_n is the sum of n independent c_0-valued r.v.'s, each of which takes two values and each of which has expected value 0. Clearly

$$|f_n(s)| \equiv 1 \quad \text{and} \quad E|f_n| = \|f_n\|_1 = 1\,.$$

But $f_n(s)$ does not converge strongly in c_0, or even in the more comprehensive space l^∞, at any irrational point s. On the other hand, since $(c_0)^* = l^1$, and since the sequence $\langle f_n(s),y\rangle$ converges for every s, for any $y \in l^1 = (c_0)^*$, the sequence $f_n(s)$ converges weakly but not to any element of c_0. Further, since $l^\infty = (l^1)^*$, it follows that a martingale f_n taking values in a space $X = (Y)^*$, may converge to f_∞ in the weak *-topology of X (that is, the Y topology of X) without being strongly or weakly convergent. The last remark is verified by noting that

$$f_\infty(s) = (\gamma_1(s),\ldots,\; \gamma_n(s),\ldots)$$

has a non-separable range in l^∞.

It is to be noted, however, that for any sequence a_n tending to 0, no matter how slowly, the series $\sum a_n\gamma_n(s)e_n$ of c_0-valued independent r.v.'s converges everywhere unconditionally but not absolutely if $\sum|a_n| = +\infty$. But $E|a_n\gamma_n(s)e_n|^2 = |a_n|^2$ so that the variance series may be chosen to diverge. Thus one may have a c_0-valued sequence of independent r.v.'s Y_n which are uniformly bounded and of zero expectation and such that ΣY_n converges a.e. (even unconditionally) without the variance series being convergent, in contradiction to a known theorem in the scalar-valued case. I hope to pursue this matter further in other publications. The example above may also be looked at as the martingale version of a counter-example of Clarkson [7, p. 414], of an l^∞-valued function of bounded variation which is nowhere differentiable although it satisfies a Lipschitz condition.

Note added in proof (received October 14, 1968).

In Section 3 the following lemma is needed to complete the discussion: If (S, Σ, P) is a probability space and μ is a σ-additive, X-valued, P-absolutely continuous function of bounded variation defined on a sub-σ-algebra Σ_0, then μ has an extension $\tilde{\mu}$ to Σ which is also σ-additive, P-absolutely continuous and of bounded variation.

Indeed an extension $\tilde{\mu}$ is defined by the formula

$$\tilde{\mu}(A) = \int P(A|\Sigma_0)\, d\mu$$

where $P(A|\Sigma_0)$ is the conditional probability of A given Σ_0 and the integral with respect to μ is taken in the sense of, say, [10, p. 323]. This simple extension came out of a discussion with Professor J. Neveu; my original one was a bit more complicated.

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