ON THE STRUCTURE OF THE ORTHOGONAL GROUP

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Let $V$ be a vector space over a field $k$ of characteristic $\neq 2$, let $Q$ be a non-degenerate quadratic form on $V$. Let $G=O(V,Q)$ denote the group of all isometries of $V$. We shall place a topology on $O(V,Q)$ such that $O(V,Q)$ becomes a topological group which is discrete if and only if $V$ is finite-dimensional. Further, the classical structure theorems for $O(V,k)$ in the finite-dimensional case carry over to the infinite-dimensional situation for the topological group $O(V,k)$ (see [1, Chapter V]).

1. Preliminaries.

Let, then, $(V,Q)$ be an arbitrary vector space over a field of characteristic $\neq 2$ with non-degenerate quadratic form $Q$. Call such a space a $Q$-space hereafter. Let $G=O(V,Q)$ be the orthogonal group of all isometries of $V$. Let $\mathcal{E}_0, \mathcal{E}$ be respectively the set of all finite dimensional non-singular subspaces of $V$, and the set of all subspaces of $V$. Let $\mathcal{H}$ be the set of all subgroups of $G$.

We define two maps: $\varrho: \mathcal{E} \to \mathcal{H}$, $\chi: \mathcal{H} \to \mathcal{E}$ as follows: If $E \in \mathcal{E}$, $H \in \mathcal{H}$, then

$$\varrho(E) = \{\sigma \in G \mid \sigma|_E = \text{identity on } E\}$$

and

$$\chi(H) = \{x \in V \mid \sigma(x) = x \text{ for all } \sigma \in H\}.$$

Some of the properties of these two maps are:

(i) If $E_1 \subseteq E_2$, $H_1 \subseteq H_2$, $E_i \in \mathcal{E}$, $H_i \in \mathcal{H}$, then $\varrho(E_1) \supseteq \varrho(E_2)$, $\chi(H_1) \supseteq \chi(H_2)$, that is, $\varrho$, $\chi$ are order reversing.

(ii) If $E \in \mathcal{E}$, $H \in \mathcal{H}$, then $\chi \circ \varrho(E) \supseteq E$ and $\varrho \circ \chi(H) \supseteq H$.

(iii) If $E_1 \in \mathcal{E}_0$, $E_2 \in \mathcal{E}$, $E_1 \subset E_2$, then $\varrho(E_1) \supseteq \varrho(E_2)$.

(iv) If $E \in \mathcal{E}$, $H \in \mathcal{H}$, then $\varrho \circ \chi \circ \varrho(E) = \varrho(E)$ and $\chi \circ \varrho \circ \chi(H) = \chi(H)$.

(v) If $E \in \mathcal{E}_0$, then $\chi \circ \varrho(E) = E$, so $\chi \circ \varrho|_{\mathcal{E}_0} = 1_{\mathcal{E}_0}$.

(vi) Let $\sigma \in G$, $E \in \mathcal{E}$, then $\varrho(\sigma(E)) = \sigma \varrho(E) \sigma^{-1}$.

**Proof.** We shall prove only (iii) and (v); all the others are more or less obvious.

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(iii): Let \( V = E_1 \perp E_1^\ast \), where \( E_1^\ast \) denotes the space \( \{ y \in V \mid y \perp E \} \) for any space \( E \in \mathcal{E} \), so that \( E_2 \supset E_1 \) implies the existence of an \( x \in E_2 \) such that \( x \in E_1^\ast \). Let \( \sigma = 1_{E_1} \perp (-1_{E_1^\ast}) \), then \( \sigma \in G \) and \( \sigma(x) = -x \). Also \( \sigma|_{E_1} = \text{identity on } E_1 \), so \( \sigma \circ \varrho(E) \). 

(v): Now \( \chi \circ \varrho(E) \supseteq E \) for all \( E \in \mathcal{E} \) and \( \varrho \circ \chi \circ \varrho(E) = \varrho(E) \) by (iv). Hence, if \( \chi \circ \varrho(E) \supseteq E \in \mathcal{E}_0 \), then \( \varrho(\varrho(E)) \subseteq \varrho(E) \) contradicting (iv).

2. The Topology on \( O(V) = G \).

Let \( \mathcal{U} = \{ U \mid U = \varrho(E), E \in \mathcal{E}_0 \} = \varrho(\mathcal{E}_0) \). Then we have:

(i) If \( U_1, U_2 \in \mathcal{U} \), there is a \( U \in \mathcal{U} \) such that \( U \subseteq U_1 \cap U_2 \).

(ii) If \( U \in \mathcal{U} \), then \( U = U, U^{-1} = U \).

(iii) If \( U \in \mathcal{U} \), \( \sigma \in G \), then there is a \( U' \in \mathcal{U} \) such that \( \sigma U' \subseteq U \).

(iv) \( \bigcap_{U \in \mathcal{U}} U = \{ 1_V \} \).

**Proof.**

(i) Let \( U_i = \varrho(E_i), E_i \in \mathcal{E}_0 \), then

\[
U_1 \cap U_2 = \varrho(E_1) \cap \varrho(E_2) = \{ \sigma \in G \mid \sigma(x) = x \text{ for } x \in E_1 \cup E_2 \}.
\]

Let \( E \in \mathcal{E}_0 \) be such that \( E \supseteq E_1 \cup E_2 \). Then

\[
U = \varrho(E) \subseteq \varrho(E_1) \cap \varrho(E_2) = U_1 \cap U_2.
\]

(ii) This is obvious.

(iii) Let \( U = \varrho(E), E' = \sigma^{-1}(E), U' = \varrho(E') \), then \( E' \in \mathcal{E}_0 \) as \( E \in \mathcal{E}_0 \). But then

\[
\varrho(\sigma(E')) = \varrho(E) = U = \sigma(\varrho(E'))\sigma^{-1} = \sigma U' \sigma^{-1}.
\]

(iv) Let \( \sigma \in \bigcap_{U \in \mathcal{U}} U \), then \( \sigma \in \varrho(E) \), for all \( E \in \mathcal{E}_0 \), hence \( \sigma(x) = x \) for all \( x \in E \), for all \( E \in \mathcal{E}_0 \). But \( \bigcup_{E \in \mathcal{E}_0} E = V \). Hence \( \sigma = 1_V \).

These facts imply that \( \mathcal{U} \) may be taken as a fundamental system of neighborhoods of the identity for a Hausdorff topology on \( G = O(V, Q) \). We shall call this the finite topology (see [3]).

**Theorem 1.** \( G = O(V, Q) \) with the finite topology, is discrete if, and only if, \( V \) is finite-dimensional.

**Proof.** If \( V \) is finite-dimensional, then clearly \( G \) is discrete, \( \{ 1_V \} \) is open and, hence, there is a \( U \in \mathcal{U} \) such that \( U \subseteq \{ 1_V \} \), so

\[
U = \varrho(E) = \{ 1_V \} \quad \text{for some } E \in \mathcal{E}_0.
\]

Let \( x \in V \) and \( E_x \in \mathcal{E}_0 \) be such that \( E_x \supseteq \{ x \} \cup E \). Then
\[ \varrho(E_x) \subseteq \varrho(E) = \{1_E\}, \]
so \( E_x = E \) by (iii) of section 1. As \( x \) was arbitrary, \( E = V \). This completes the proof.

If \( E \in \mathcal{E}_0 \), then \( \varrho(E) \) is, of course, an open subgroup and, consequently, closed. But we have more generally

**Proposition 1.** Let \( E \in \mathcal{E} \), and let \( G = O(V) \) have the finite topology, then \( H = \varrho(E) \) is closed and for any \( H \in \mathcal{H} \), \( \varrho(H) \subseteq \overline{H} \). Hence \( \varrho \) maps \( \mathcal{E} \) onto the set \( \mathcal{H} \) of all closed subgroups of \( G \).

**Proof.** Let \( \sigma \in \overline{H} \), \( H \in \mathcal{H} \). Then for any \( E_0 \in \mathcal{E}_0 \), we must have \( \varrho(E_0) \sigma \cap H \neq \emptyset \). If \( x \in E \subseteq \chi(H), x \neq 0 \), let \( E_x \in \mathcal{E}_0 \) be such that \( x \in E_x \).

Then there is an \( \eta_x \in \varrho(E_x) \) such that \( \eta_x^{-1} \sigma \in H \), so that \( \eta_x^{-1} \sigma |_E = 1_E \).

In particular, \( \sigma(x) = \eta_x(x) = x \), as \( \eta_x \in \varrho(E_x) \). Thus \( \sigma(x) = x \), for all \( x \in E \); hence \( \sigma \in H \). So \( H = \overline{H} \), the closure of \( H \). Next, since \( H \subseteq \varrho(\chi(H)) \) is closed, \( \overline{H} \subseteq \varrho(\chi(H)) \). This completes the proof.

Notice that if \( A = \{ \pm 1_v \} \), then \( A = \overline{A} \subset O(V) = \varrho(\chi(A)) \).

**Proposition 2.** If \( (V,Q) \) is infinite dimensional, then \( O(V,Q) \) is a totally disconnected non-locally compact group.

**Proof.** Let \( E_1 \in \mathcal{E}_0 \) be such that \( \dim E_1 \geq 2 \), then \( V = E_1 \perp E_1^* \), \( \varrho(E_1) \) is an open subgroup of \( G \) and, hence, is closed. Further, \( \bigcap_{E \in \mathcal{E}_0} \varrho(E) = \{1_E\} \), so \( O(V) = G \) is totally disconnected.

In order to prove that \( G \) is not locally compact, it suffices to show that \( G \) itself is non-compact since \( \varrho(E') = G' \) compact would imply that \( O(E'^*, Q|_{E'^*}) \), the finite topology of which is the inherited topology from \( O(E, Q) \), would be compact. So we let \( G_1 = \varrho(E_1), \dim E_1 \geq 2 \), \( E_1 \in \mathcal{E}_0 \) and consider the left coset space \( O(V,Q)/G_1 = G/G_1 \). Suppose that \( G \) is compact, then the natural map \( \pi: G \to G/G_1 \) shows that \( G/G_1 \) is compact. But \( G_1 = \varrho(E_1) \) is open and, hence, \( G_1 = \sigma \in G/G_1 \) is also open. Thus \( G/G_1 \) is discrete and, therefore, finite. Consider next two cases:

(i) \( k \) infinite. The elements of \( O(E_1) \) may be identified with the elements of the subgroup \( O(E_1) \perp 1_{E_1^*} \) of \( G \). Denote this group by \( G_2 \).

Then \( \tau, \tau' \in G_2, \tau \equiv \tau \mod G_1 \) if, and only if, \( \tau^{-1} \tau \in G_1 \). But \( G_2 \cap \varrho(E_1) = \{1_E\} \), so \( \tau = \tau' \). Therefore, since \( G_2 \) is infinite, \( G/G_1 \) is an infinite set. This contradicts compactness; hence \( G \) is not compact.

(ii) If \( k \) is finite, then we use lemma 1, proved in the next section, to provide us with a sequence of subspaces \( F_1, F_2, \ldots \), such that \( F_i \perp F_j, j \neq i \), and \( F_i \) are hyperbolic planes. Then construct the following maps \( \sigma_i \) such that \( \sigma_i: F_i \to F_i \) isometrically, and extend \( \sigma_i \) to \( V \) (see [4]). Choose \( E_1 \) to equal \( F_1 \). Then \( \pi: \sigma_i \to \overline{\sigma_i} \), and
\[ \bar{\sigma}_i = \bar{\sigma}_j \iff \sigma_j^{-1} \sigma_i \in G_1 = \varrho(E_1). \]

For if \( i \neq j \),
\[
\sigma_j^{-1} \sigma_i : E_1 \overset{\varrho}{\rightarrow} F_\iota \overset{\sigma_j^{-1}(F_\iota)}{\rightarrow} E_1,
\]
hence \( \sigma_j^{-1} \sigma_i \notin G_1 \). Thus again we have an infinite set \( G/G_1 \) which contradicts compactness.

Let \( G_F = \{ \sigma \in G = O(V) \mid \sigma \in \varrho(E^*), E \in \varepsilon_0 \} = \) group generated by all \( \sigma \in G \) such that \( \sigma \) leaves \( E^* \) elementwise fixed, where \( E \in \varepsilon_0 \), \( V = E \perp E^* \). We have

**Theorem 2.** Using the finite topology on \( O(V) \), we find that the group \( G_F \) is dense in \( O(V) \), that is, \( \bar{G}_F = O(V) \).

**Proof.** Let \( \sigma \in G = O(V) \). We have to show that \( \sigma \in \bar{G}_F \), that is, for all \( E \in \varepsilon_0 \), there is a \( \eta \in \varrho(E) \) such that \( \sigma \eta \in G_F \), or that there is a \( \tau \in G_F \) such that \( \sigma^{-1} \tau = \eta \in \varrho(E) \). Let \( E^\sigma = \sigma(E) \), then let \( E_1 \) be a finite dimensional semi-simple subspace of \( V \) such that \( E_1 \cong E + E^\sigma \). Now \( V = E_1 \perp E_1^* \) and \( \sigma : E \rightarrow E^\sigma \), with \( E, E^\sigma \) both contained in \( E_1 \); hence, by Witt's Theorem, we can extend \( \sigma \) to \( \sigma' \) in \( O(E_1) \). Let \( \tau = \sigma' \perp 1_{E_1^*} \), then clearly \( \tau \in G \) and \( \tau \in G_F \). Further, \( \sigma^{-1} \tau|_E = \sigma^{-1} \sigma|_E = 1_E \); hence \( \sigma^{-1} \tau \in \varrho(E) \). This proves the theorem.

3. The structure of \( O(V, Q) \).

In this section we wish to generalize, to infinite dimensional \( Q \)-spaces, a number of results which will aid us in considering further the structure of the orthogonal group \( O(V) \). Some we state without proof. Many times we will use the notation: \( f_Q(x, y) = x \cdot y \), for \( x, y \in V, f_Q \) the associated bilinear form.

**Lemma 1.** Let \( (V, Q) \) be a semi-simple \( Q \)-space over \( k \). If \( \dim V \geq 2 \) and \( k \) is finite, then for any \( a \in k \) there is a \( x \in V \) such that \( f_Q(x, x) = a \). If \( \dim V \geq 1 \) and \( V \) has isotropic vectors, then when \( k \) is any field and for any \( a \in k \) we can find \( x \in V \) so that \( f_Q(x, x) = a \).

**Proof.** For infinite dimensional \( V \) the proof is as in the finite case (see e.g. [1]).

**Corollary.** If \( (V, Q) \) is as in the lemma with \( \dim V \geq 3 \), then \( V \) contains isotropic vectors if \( k \) is finite.

Let \( Z(V, Q) \) denote the center of \( O(V, Q) = G \) and \( O(V, Q)' \) the group theoretic commutator subgroup of \( O(V, Q) \). Let \( \Omega_F(V, Q) = (G_F)' \), where \( G_F \) was defined in section 2. Let \( \Omega(V, Q) = O(V, Q)' \), the closure of the group generated by the commutators.
Lemma 2. Let $(V, Q)$ be a semi-simple $Q$-space, $S$ a two dimensional non-singular (semi-simple) subspace. Unless $S$ is a hyperbolic plane and $k = GF(3)$, there is a map $\sigma \in \Omega_F(V, Q)$ such that $S^* = \{x \in V \mid \sigma(x) = x\}$.

Proof. It is well known that there is a $\sigma' \in \Omega(S, Q)$ such that $\sigma'(x) + x$ for all $0 \pm x \in S$. Define $\sigma$ as: $\sigma = \sigma' \perp 1_{S^*}$. $(V = S \perp S^*)$, then $\sigma: V \to V$ and is clearly in $\Omega_F(V, Q)$. This $\sigma$ is the element that works.

Lemma 3. If $(V, Q)$ is a semi-simple $Q$-space with dimension $\geq 3$, let $\sigma \in O(V, Q)$ be such that $\sigma \tau = \tau \sigma$ for all $\tau \in \Omega_F(V, Q)$.

Then $\sigma = \pm 1_V$.

Proof. First we show that if $f_Q(x, x) \not= 0$, then $\sigma(x) = ax$, $a \in k$, $a = \pm 1$.

a) Assume $k = GF(3)$. Let $x$ be such that $f_Q(x, x) \not= 0$, let $S(x)$ be the space of $x$. So $V = S(x) \perp S^*(x)$ and dim $S(x^*) \geq 2$, so that there is an element $y \in S(x^*)$ such that $f_Q(y, y) \not= 0$, $x \perp y$. Let $S = S(x, y)$ be the span of $x$ and $y$, $S$ is then semi-simple. Lemma 2 implies the existence of $\varphi \in \Omega_F(V, Q)$ such that $S^* = \{z \in V \mid \varphi(z) = z\}$. Let $z \in S^*$, 

$$\varphi(\sigma(z)) = \sigma \varphi(z) = \sigma(z),$$

hence $\sigma(z) \in S^*$ for any $z \in S^*$; so that $\sigma(S^*) \subseteq S^*$. Similarly, since $\sigma^{-1} = \varphi^{-1} \sigma^{-1}$ for all $\varphi \in \Omega_F(V, Q)$, we get that $\sigma^{-1}(S^*) \subseteq S^*$, and hence $\sigma(S^*) = S^*$. Consequently $\sigma(S) = S$. Now dim $S^* \geq 1$ and is semi-simple; so that there is a $y' \in S^*$ such that $f_Q(y', y') \not= 0$. Apply the same argument above to $S' = S(x, y')$ and get that $\sigma(S') = S'$. Hence

$$\sigma(S \cap S') = S \cap S' = S(x);$$

so that $\sigma(x) = ax$ ($a = \pm 1$).

b) Let $k = GF(3)$, hence $k^* = \pm 1$. By lemma 1, since dim $V \geq 2$, for any $a \in k$ there is a vector $x \in V$ such that $f_Q(x, x) = a \in k^*$. So $V = S(x) \perp S(x^*)$, dim $S(x^*) \geq 2$, and hence, again there is a $y \in S(x^*)$ such that $f(y, y) = a$. Put $S = S(x, y)$. If $z = ax + \beta y$, then

$$z \cdot z = \alpha^2 a + \beta^2 a = a(\alpha^2 + \beta^2),$$

so

$$f(z, z) = 0 \iff \alpha = 0 \text{ and } \beta = 0 \text{ or } z = 0.$$

Hence $S$ is anisotropic and not a hyperbolic plane, so that, by lemma 2, there is a $\varphi \in \Omega_F(V, Q)$ such that $S^* = \{z \mid \varphi(z) = z\}$. Then we get as before $\sigma(S) = S$.

i) If dim $V \geq 4$, then dim $S^* \geq 2$. The same method as in a) yields $\sigma(x) = c \cdot x$. 
ii) If \( \dim V = 3 \), then \( \dim S^* = 1 \). Take \( y' \in S^*, y' \neq 0 \); semi-simplicity of \( S^* \) implies that \( f(y', y') \neq 0 \). Hence \( f_Q(y', y') = \frac{\pm a}{a} \in k^* \). If \( f_Q(y', y') = +a \), apply a previous argument to get that \( \sigma(x) = c \cdot x \). If \( f_Q(y', y') = -a \), then \( V = S(x, y, y') \). The vectors \( x \pm y, y' \) are orthogonal,

\[
f_Q(x, x) = f_Q(y, y) = a = -f_Q(y', y'),
\]

and with \( \varepsilon = \pm 1 \)

\[
f_Q(x + \varepsilon y, x + \varepsilon y) = f_Q(x, x) + f_Q(y, y) = 2a = -a = f_Q(y', y').
\]

Therefore \( S(x \pm y, y') \) is anisotropic, and by the previous argument we get

\[
\sigma(S(x \pm y, y')) = S(x \pm y, y').
\]

Hence

\[
\sigma(S(x, y) \cap S(x \pm y, y')) = S(x, y) \cap S(x \pm y, y') = S(x \pm y);
\]

so we get \( \sigma(x + \varepsilon y) = c_i(x + \varepsilon y) \), \( i = 1, 2 \). But \( (x + y) \perp (x - y) \), and \( x \pm y, y' \)

is an orthogonal basis of \( V \); so that \( \sigma(y') = cy' \), hence \( \sigma(x) = c \cdot x \).

Take now any finite dimensional subspace \( S \) of \( V \). Let \( S' \) be a finite dimensional semi-simple subspace of \( V \) containing \( S \). Let \( x_1, \ldots, x_n \) be an orthogonal basis of \( S' \), then \( \sigma(x_i) = c_i x_i \).

So: a) If \( f_Q(x_1 + x_2, x_1 + x_2) \neq 0 \), then

\[
\sigma(x_1 + x_2) = c(x_1 + x_2) = c_1 x_1 + c_2 x_2;
\]

so that \( c = c_1 = c_2 \).

b) If \( f_Q(x_1 + x_2, x_1 + x_2) = 0 \), then

\[
Q(x_1 + x_2 + x_3) = Q(x_1 + x_2) + Q(x_3) = Q(x_3) \neq 0 \quad \dim V \geq 3.
\]

So that

\[
\sigma(x_1 + x_2 + x_3) = c(x_1 + x_2 + x_3) = \sum_{i=1}^{3} c_i x_i,
\]

and hence \( c = c_i, i = 1, 2, 3 \).

Thus we get \( c_1 = c_2 = \ldots = c_n = \pm 1 \). Then clearly from this we get that \( \sigma = \pm 1_V \). This completes the proof.

**Corollary.** \( Z(V, Q) = \{ \pm 1_V \} \), when \( (V, Q) \) is a semi-simple \( Q \)-space of \( \dim \geq 3 \).

For every finite dimensional semi-simple \( Q \)-space \( (W, Q) \) with associated bilinear form \( g_Q \) we have from the finite dimensional theory a homomorphism \( \theta_w \) called the spinorial norm, \( \theta_w : O^+(W) \to k^* / k^{*2} \), where \( O^+(W) \) is the group of rotations in \( O(W) \). This map is defined as follows: if \( \sigma = \tau_{x_1} \tau_{x_2} \ldots \tau_{x_r} \), where \( \tau_x \) is the symmetry defined by \( x \), that is, if

\[
\tau_x(y) = y - 2 \frac{x \cdot y}{x \cdot x} x,
\]
then
\[ \theta_w(\sigma) = x_1^2 x_2^2 \ldots x_r^2 \text{ for } x_i = g_Q(x, x). \]

Some of the properties of the spinorial norm for \((W, g)\) are as follows:

i) \(\Omega(W, g_Q) \subseteq \ker \theta_w\),

ii) \(\theta_w\) maps \(O^+(W)\) onto \(k^*/k^{*2}\), when \((W, g_Q)_Q\) has isotropic vectors.

iii) If \(W\) contains non-zero isotropic vectors, then \(\Omega(W, g) = \ker \theta_w\).

iv) If \(W = U \perp V\), then \(\ker \theta_U = O^+(U) \cap \ker \theta_w\).

v) Suppose \(\sigma = \tau_x \cdot \tau_y\),

\[ \tau_x(u) = u - 2 \frac{u \cdot x}{x \cdot x} x. \]

\(\sigma \in \ker \theta_w\), then \(\sigma \in \Omega(W, g_Q)\). If \(\dim W = 2\), then \(\ker \theta_w = \Omega(W, g_Q)\) and each element of \(\Omega(W)\) is a square of a rotation. If \(\dim W = 3\), then \(\ker \theta_w = \Omega(W)\) and each element of \(\Omega(W)\) is the square of a rotation with the same axis.

We set down the following notation: If \(U\) is a finite dimensional semi-simple subspace of \((V, Q)\) and \(\tau \in O(U)\) we shall hereafter identify \(\tau\) with \((\tau \perp 1_{U\perp}) \in O(V)\).

**Lemma 4.** Let \((V, f)\) be any \(Q\)-space. Suppose that \(k\) is a finite field and \(\tau\) is an isometry of a subspace \(U_1\) of \(V\) onto a finite dimensional subspace \(U_2\) of \(V\). If \(U_2\) is contained in a finite dimensional semi-simple \(Q\)-space \(W\) of \(V\) with \(\text{codim} W \geq 2\), then we can extend \(\tau\) to an element \(\varphi \in \Omega_F(V, Q)\).

**Proof.** Let \(W_2\) be a finite dimensional semi-simple \(Q\)-subspace of \(V\) containing \(U_1, U_2\) and \(W\) such that \(\dim (W_2/W) \geq 2\). Then we can extend \(\tau\) to a rotation \(\sigma\) of \(W_2\), since we can always multiply by a reflection of \(W^\perp \cap W_2\). Now if \(\varphi_1\) is a rotation of \(W^\perp \cap W_2\), then \(\varphi_1 \cdot \sigma\) also extends \(\tau\). But, since \(\dim (W^\perp \cap W_2) \geq 2\) and \(k\) is finite, we know \(W^\perp \cap W_2\) contains elements with arbitrary squares. Hence there is a rotation \(\varphi_1 \in O(W^\perp \cap W_2)\) such that \(\theta_{w_2}(\varphi_1) = \theta_{w_2}(\sigma)\), thus \(\theta_{w_2}(\varphi_1 \cdot \sigma) = 1\), and hence \(\varphi_1 \cdot \sigma \in \Omega(W_2, Q|W_2)\). Since \(\dim W_2 \geq 3\), then \(W_2\) contains non-zero isotropic vectors; so that \(\Omega(W_2, Q|W_2') = \ker \theta_{w_2}\). Let \(\varphi = \varphi_1 \cdot \sigma \perp 1_{w_2'}\), then \(\varphi \in \Omega_F(V, Q)\).

**Corollary.** If \(k\) is finite, \(\dim V \geq 4\), and \(x, y\) are non-zero isotropic vectors, then there is a \(\lambda \in \Omega_F(V, Q)\) such that \(\lambda x = y\).

**Proof.** \(x\) is in some hyperbolic plane, as is \(y\). Let \(\tau\) map the first hyperbolic plane onto the other isometrically and such that \(\tau x = y\). Then extend \(\tau\) by the lemma.
Lemma 5. Let $U_1, U_2$ be semi-simple isometric subspaces of $(V,Q)$, $(V,Q)$ a semi-simple $Q$-space. Suppose that $U_2$ is finite dimensional and contains isotropic lines. Then there is a $\lambda \in \Omega_F(V,Q)$ such that $U_2 = \lambda(U_1)$.

Proof. Let $W_1$ be a semi-simple finite dimensional subspace containing $U_1$ and $U_2$. By multiplying by an appropriate symmetry, we can find a rotation $\sigma$ of $W_1$, such that $\sigma U_1 = U_2$. Again, we can follow this by any rotation $\varphi$ of $U_2$. By Lemma 1, we can achieve: $\theta_{w_1}(\varphi) = \theta_{w_1}(\sigma)$, since $U_2$ has isotropic lines. Hence $\lambda_1 = \varphi \cdot \sigma$ is such that $\theta_{w_1}(\lambda_1) = 1$, so that $\lambda_1 \in \Omega(W_1)$ and $U_2 = \lambda(U_1)$. Then $\lambda = \lambda_1 \perp 1_{w_1 \star}$ is the desired map.

Corollary. If $\dim V \geq 3$ and $x, y$ are non-zero isotropic vectors, then there is an element $\beta \in k^*$ such that for any $\alpha \in k^*$, there is an element $\lambda \in \Omega_F(V,Q)$ such that $\lambda(x) = \beta \alpha^2 y$.

Proof. Let $y, y'$ be a hyperbolic pair, and let $W$ be a semi-simple subspace of finite dimension, such that $x, y, y'$ are in $W$. Choose a rotation $\sigma \in O(W)$ such that $\sigma x = y$ (possible since $\dim V \geq 3$). Let $\theta_w(\sigma) = \beta k^{\star2}$, and let $\varphi$ be a rotation of $S(y, y')$ such that $\varphi(y) = \beta \alpha^2 y$. Then $\theta_w(\varphi) = \theta_w(\sigma)$ and $\lambda = \varphi \sigma$ is the desired element of $\Omega_F(V,Q)$ (identified with $1_{w_1 \star \perp \lambda}$).

Lemma 6. Let $(V,f)$ be a semi-simple $Q$-space with $\dim V \geq 5$. Let $P = S(x, y)$ be a singular plane (that is, $\text{rad } P \neq \{0\}$) where $x^2 = y^2 = 0$. Then there is $\lambda \in \Omega_F(V,Q)$ such that $\lambda y = y$, $\lambda x = z$, where $S(x, z)$ is semi-simple.

Proof. Let $S(u)$ be the radical of $P$, then $P = S(y, u)$ and $x = ax + \beta u$, $\beta \neq 0$. But $x^2 = y^2$, hence

$$x^2 = ax^2 + 2\beta xy + \beta^2 u^2 = ax^2 y^2.$$ 

So that $a = \pm 1$, and we can replace $xy$ by $y$, $\beta u$ by $u$, to obtain $x = y + u$. Let $H = S(y)^*$ be the hyperplane orthogonal to $y$, since $u \in H$, there is an isotropic vector $v \in H$, such that $u \cdot v = -x^2$. Then we have two cases:

a) $k$ is a finite field: $\dim H \geq 4$, so that there is a $\lambda_1 \in \Omega_F(H)$ such that

$$\lambda_1 u = v, \quad \lambda = 1_H \perp \lambda_1 \in \Omega_F(V,Q).$$

Hence

$$\lambda(y) = y, \quad \lambda x = \lambda(y + u) = y + v = z,$$

and

$$x \cdot z = (y + u) \cdot (y + v) = y^2 + y \cdot v + y \cdot u + u \cdot v = 0,$$

$$z \cdot z = x^2 = y^2 \neq 0.$$ 

Therefore $S(x, z)$ is semi-simple.
b) $k$ is an infinite field; thus there is a $\lambda \in \Omega_F(H, f|H)$ such that

$$\lambda u = \alpha \beta^2 v, \; \lambda y = y,$$

$\alpha, \beta$ chosen as follows:

$$\lambda x = \lambda(y + u) = y + \alpha \beta^2 v = z.$$ 

Then

$$z^2 = y^2 = x^2,$$

$$x \cdot z = x \cdot y + \alpha \beta^2 x \cdot v = (y + u) \cdot y + \alpha \beta^2 (y + u) v = y^2 - \alpha \beta^2 x^2.$$ 

Then $S(x, z)$ is non-singular when we can choose $\beta$ such that

$$x \cdot z = y^2 - \alpha \beta^2 x^2 \pm x^2.$$

Since $k$ is infinite, one can clearly find an appropriate $\beta$ and $\alpha$.

**Lemma 7.** Let $(V, Q)$ be a semi-simple $Q$-space, $x \in V$ such that $f_Q(x, x) \neq 0$. Suppose $\sigma$ is an isometry of $V$ which keeps fixed every line $L$ generated by a vector $y$ such that

$$y^2 = f_Q(y, y) = x^2.$$ 

Then $\sigma = \pm 1_V$ if $\dim V \geq 4$ for any $k$.

**Proof.** Since $f_Q(\sigma x, \sigma x) = f_Q(z, x) \neq 0$, we may assume that $\sigma x = x$ by replacing $\sigma$ by $-\sigma$ if necessary. Let $H = S(x)^*$. We have two cases:

a) $H$ contains non-zero isotropic vectors: Let $u \in H$ be such that $u^2 = 0, \; u \neq 0$. Since $(x + u)^2 = x^2$, it follows that $\sigma(x + u) = \varepsilon(x + u)$ where $\varepsilon = \pm 1$. But

$$\sigma(u) = \sigma(x + u - x) = \varepsilon(x + u) - x$$

is isotropic, hence $\varepsilon = +1$. So

$$\sigma(u) = (x + u) - x = u.$$ 

Let $y \in H$. Then $y$ is in some hyperbolic plane of $H$. But $\sigma$ equals the identity on any hyperbolic plane of $H$ so that $\sigma(y) = y, \; \sigma(x) = x$ implies that $\sigma = 1_V$, hence the original $\sigma = \pm 1_V$.

b) $H$ is anisotropic. Let $z \neq 0, \; z \in H$. If there are at least six rotations of the plane $P = S(x, z)$, then they carry $S(x)$ into three distinct lines which are kept fixed by $\sigma$, which gives us, by use of the properties of $\sigma$ and $\sigma x = x$, that $\sigma z = z$. But if $k$ is finite, then since $\dim H \geq 3$, by the corollary to lemma 1, $H$ has non-zero isotropic vectors, so that we are in case a). And if $k$ is infinite, then it is known that there are more than six rotations of the plane $P$ when $P$ is either hyperbolic or anisotropic. Of course, $P$ is semi-simple.

This completes the proof.
Now we generalize to infinite dimensions a proposition which is practically the result we want (see [1]).

**Proposition 3.** Suppose \((V,Q)\) is a semi-simple \(Q\)-space with \(\dim V \geq 5\). Let \(H\) be a subgroup of \(O(V)=G\) which enjoys the following properties:

a) \(H\) is invariant under transformation by elements of \(\Omega_{F}(V,Q)\); that is, if \(\sigma \in \Omega_{F}(V,Q)\), then \(\sigma H \sigma^{-1} \subseteq H\).

b) \(H\) is not contained in the center of \(G=O(V)\).

Then \(H\) will contain an element \(\sigma \pm 1_{V}\) which is the square of a three dimensional rotation; that is, it is the square of a rotation arising from a three-dimensional space.

**Proof.** Pick \(\sigma \in H\) such that \(\sigma \pm 1_{V}\) (corollary to lemma 3). Then \(\sigma\) must move some non-isotropic line \(S(x)\), otherwise \(\sigma = \pm 1_{V}\), by lemma 7. Now we can choose this \(\sigma\) such that the plane \(P=S(x,\sigma(x))\) is semi-simple, as follows: Suppose \(P\) is singular. Using \(\sigma(x)^{2}=x^{2}\), it follows by lemma 6 that, there is a \(\lambda \in \Omega_{F}(V)\) such that \(\lambda(\sigma(x))=\sigma(x), \lambda x=zx\), with \(S(x,z)\) semi-simple. Take \(\varrho=\lambda \sigma^{-1} \lambda^{-1} \sigma\). This is in \(H\), by hypothesis a), and \(\varrho(x)=z\), hence \(S(x,\varrho(x))\) is semi-simple. Let \(\sigma\) be this \(\varrho\). So we may assume that \(P=S(x,\sigma(x))\) is a semi-simple plane. Using this \(\sigma\) we claim that there is a \(\varrho \pm 1_{V}\) of \(H\) which keeps some non-isotropic line fixed. To see this we may assume that \(\sigma\) moves every non-isotropic line, otherwise take \(\varrho=\sigma\). Suppose there were a \(\lambda \in \Omega_{F}(V,Q)\) such that it keeps every vector of \(P\) fixed and does not commute with \(\sigma\), then \(\varrho=\lambda^{-1} \sigma^{-1} \lambda\).

\(\sigma \in H\), by hypothesis a) and \(\varrho(x)=\lambda^{-1}(x)=x\). Thus \(\varrho \pm 1_{V}\), and \(\lambda \sigma \varrho \sigma\lambda\) implies that \(\varrho \pm 1_{V}\). Thus we would have the desired \(\varrho\). To prove the existence of the above \(\lambda\) consider the cases:

i) \(\sigma P \neq P\). Let \(u \in P\) such that \(\sigma u \notin P\), so that \(\sigma u = v + w, \ v \in P, \ w \in P^{*}\), with \(w \neq 0\). But the \(\dim P^{*} \geq 3\). Then one can pick a three dimensional semi-simple subspace of \(P^{*}\) containing \(w\), say \(W\), and then find \(\lambda \in \Omega(W)\) which moves \(w\) (see [1, p. 105]). Extend \(\lambda\) to \(V\), and get

\[\lambda \sigma(u) = \lambda(v+w) = v + \lambda(w) \pm v + w = \sigma u = \sigma \lambda(u),\]

hence \(\lambda \sigma \pm \lambda \lambda \in \Omega_{F}(V)\).

ii) \(\sigma P = P\). Then \(\sigma P^{*} = P^{*}\). Let \(\tau = \sigma|_{P^{*}}\). Suppose \(\varrho = \varrho \tau\) for all \(\varrho \in \Omega_{F}(P^{*})\), then by lemma 3 we know that \(\tau = \pm 1_{P^{*}}\). But this contradicts the assumption that \(\sigma\) moves the non-isotropic lines. Hence there is a \(\varrho' \in \Omega_{F}(P^{*})\) such that \(\varrho \varrho' \neq \varrho' \tau\). Extending \(\varrho'\) to \(V\), by \(\varrho = 1_{P} \perp \varrho'\), we have that \(\varrho \varrho' = \varrho \sigma, \ \varrho|_{P} = 1_{P}\).

Thus we have an element \(\sigma \pm 1_{V}\) in \(H\) such that \(\sigma\) keeps a certain non-isotropic line \(S(x)\) fixed. Then, by lemma 7, we have again that
there must be a vector \( y \) such that \( y^2 = x^2 \), but \( S(y) \) is moved by \( \sigma \). Let \( \sigma y = z \), then \( S(y) \neq S(z) \). Let \( \tau_x \) be the symmetry defined by the hyperplane \( S(x)^* \), that is

\[
\tau_x(u) = u - 2 \frac{x \cdot u}{x \cdot x} x.
\]

Let \( W \) be a semi-simple subspace of finite dimension containing \( x \) and \( y \). Then Witt’s theorem says that there is an element \( \mu \in O(W) \) and hence an element \( \mu \in O_F(V) = G_F \), such that \( \mu(x) = y \). Now \( \mu \tau_x \mu^{-1} = \tau_{\mu(x)} = \tau_y \), therefore

\[
\lambda = \tau_y \tau_x = \mu \tau_x \mu^{-1} \tau_x^{-1} \in \Omega_F(V, Q).
\]

Again, \( \sigma \tau_x \sigma^{-1} = \tau_{\sigma x} = \tau_{\pm x} = \tau_x \), and \( \sigma \tau_y \sigma^{-1} = \tau_{\sigma y} = \tau_z \). Therefore

\[
\sigma \lambda \sigma^{-1} = \sigma \tau_y \tau_x \sigma^{-1} = \sigma \tau_y \sigma^{-1} \sigma \tau_x \sigma^{-1} = \tau_z \tau_x.
\]

Hence \( \varrho = \sigma \lambda \sigma^{-1} \lambda^{-1} = \tau_z \tau_x \tau_x \tau_y = \tau_z \tau_y \). But \( \varrho \in H \), and \( \varrho \neq 1_V \), since \( S(z) \neq S(y) \). Further \( \varrho \in \Omega_F(V, Q) \), (ref. \( V \)) on the spinorial norm. Finally, let \( P = S(y, z) \), then \( \text{rad } P = P^* \cap P = \text{rad } P^* \). Since \( y^2 \neq 0 \), \( \text{dim } \text{rad } P \leq 1 \), let \( W \) be any finite dimensional semi-simple subspace of \( \text{dim } n_w \geq 5 \) containing \( P \). Then \( P^* \cap W \) contains a semi-simple subspace \( W_1 \) of dimension \( n_w - 3 \). Let \( U \) be a three-dimensional semi-simple subspace orthogonal to \( W_1 \) in \( W \). So that \( \varrho \) is a rotation of \( U \), since \( \varrho = \tau_z \tau_y \) leaves each element of \( P^* \), and hence those of \( W_1 \), fixed. But \( \varrho \in \Omega(W) \subseteq \ker \theta_u \), and by properties iv) and v) of \( \theta_u \), we get

\[
\varrho \in \ker \theta_u = O^+(U) \cap \ker \theta_u, \quad \ker \theta_u = \Omega(U).
\]

So that \( \varrho \in \Omega(U) \) and \( \varrho \) is a square of a rotation of \( U \).

This completes the proof, since \( \varrho \neq 1_V \), \( \varrho \in H \), and \( \varrho \) is the square of a rotation of \( U \) a three-dimensional semi-simple subspace of \( V \); hence \( \varrho \in \Omega_F(V, Q) \).

Suppose now that \((V, Q)\) contains isotropic vectors. Let \( U \) be a three-dimensional semi-simple subspace of \( V \), and \( \varrho \) the square of a rotation of \( U \). Suppose \( U \) is anisotropic. Then the axis of rotation of \( \varrho \) is not isotropic and thus \( \varrho \) is the square of a rotation of an anisotropic plane \( P \).

We prove that any anisotropic plane \( P \) can be imbedded in a three-dimensional semi-simple subspace \( U' \) of \( V \) containing isotropic vectors, as follows: Select \( u \neq 0 \), such that \( u^2 = 0 \) and \( u \) is not orthogonal to \( P \). This is possible as follows: Let \( S(u, v) \) be a hyperbolic plane, \( P = S(y, z) \). Then

\[
(u + \frac{1}{2} y^2 v)^2 = 2 \frac{1}{2} y^2 (u \cdot v) = y^2.
\]
Let $\tau: u + \frac{1}{2}y^2v \rightarrow y$, extend $\tau$ to $V$. Then $\tau: S(u,v) \rightarrow S(t,s)$ a hyperbolic plane containing $y$, so that

$$\tau: u + \frac{1}{2}y^2v \rightarrow t + \frac{1}{2}y^2s = y, \quad t(t + \frac{1}{2}y^2s) = \frac{1}{2}y^2 \neq 0.$$ 

Hence this $t$ works. Let $u$ then be such that $u^2 = 0$ and not $u \perp P$, set $U' = S(y,z,u)$. If $U'$ were singular, and $S(v)$ its radical, then $S(v)$ would be the only isotropic line of $U'$, since $P$ is anisotropic. So that $S(v) = S(u)$ and this implies that $u \in P^*$, contradicting the fact that not $u \perp P$. Hence $U'$ is semi-simple. This proves our assertion, so that we may assume that $\varrho$ is the square of rotation of $U$ which contains isotropic vectors in the first place. This also proves that the generators (see [1, p. 135]) $(\tau_x \tau_y)^2$ of $\Omega_F(V)$ are squares of rotations of three-dimensional subspaces $U$ which contain isotropic lines. We will use this remark below.

Now let $(V,Q)$ be a semi-simple $Q$-space of dimension $\geq 5$, containing isotropic vectors and $H$ a subgroup of $O(V)$ as in the proposition above. Let $\varrho \in H \cap \Omega_F(V)$, $\varrho \neq 1_V$,

and such that $\varrho$ is the square of a rotation of a three-dimensional semi-simple subspace $U$ containing isotropic vectors.

i) Suppose $k$ contains more than three elements. Then from the finite dimensional theory we know that $\Omega(U) \approx PSL_2(k)$, the projective special linear group, is simple. $H \cap \Omega(U)$ contains $\varrho$ and is an invariant subgroup of $\Omega(U)$, hence $\Omega(U) \subseteq H$, by simplicity of $\Omega(U)$. The subspace $U$ contains a hyperbolic plane $P$ and therefore $H$ contains $\Omega(P)$. Let $U'$ be a subspace, such as $U$, $P' \subset U'$ a hyperbolic plane of $U'$. By lemma 5, there is a $\lambda \in \Omega_F(V)$ such that $P' = \lambda P$. So that $\Omega(P') = \lambda \Omega(P) \lambda^{-1}$, hence $\Omega(P') \subseteq H$. The group $\Omega(U')$ is simple and $\Omega(U') \cap H \subseteq \Omega(P')$, a non-trivial group. But $\Omega(U') \cap H$ is invariant in $\Omega(U')$, hence $\Omega(U') \subseteq H$. This implies that $H$ contains all generators of $\Omega_F(V)$, and hence $H$ contains $\Omega_F(V,Q)$.

ii) Suppose $k = GF(3)$. We use the fact that a finite dimensional $Q$-space over a finite field contains a proper subspace of arbitrary prescribed geometry (by use of lemma 1 repeatedly). Since dim $V \geq 5$, we can find a semi-simple subspace $V'_0$ of dimension 4 which is of index 1. Then the subspace $V'_0$ contains a three-dimensional subspace $U'$ isometric to $U$, by use of the same fact. So by extending this isometry of $U'$ with $U$ to $V$, $U$ is contained in a 4-dimensional semi-simple subspace $V'_0$ of $V$ with index 1. And again by the finite dimensional theory the group $\Omega(V'_0)$ ($\approx PSL_2(kd^4)$) is simple. Further $\Omega(V'_0) \cap H$ contains $\varrho$. Hence $\Omega(V'_0) \subseteq H$. Then, as before, with $V_1$ any 4-dimensional semi-simple subspace of index 1, we have again $\Omega(V_1) \subseteq H$. If $U'$ is any sub-
space such as $U$ we can imbed it in such a space $V_1$. Thus we can get $\Omega(U') \subseteq H$. And, as in i) we thus have $\Omega_F(V) \subseteq H$. We have proved

**Theorem 3.** Suppose $(V,Q)$ is a semi-simple quadratic space of dimension $\geq 5$, and that $V$ contains isotropic vectors. Let $H$ be a subgroup of $O(V)$ having the following properties:

i) $H$ is invariant under transformation by $\Omega_F(V,Q)$; that is to say, if $\tau \in \Omega_F(V,Q)$, then $\tau H \tau^{-1} \subseteq H$.

ii) $H$ is not contained in $Z(V,Q)$.

Then $\Omega_F(V,Q) \subseteq H$.

**Corollary.** Let $(V,Q)$ be a semi-simple quadratic $Q$-space of dimension $\geq 5$, containing isotropic vectors. Let $H$ be a closed normal subgroup of $O(V,Q)$, provided with the finite topology, such that $H \not\subseteq Z(V,Q)$. Then $H$ contains the closure of $\Omega_F(V,Q)$.

If $H$ is also in $\overline{\Omega_F(V)}$, then $H = \overline{\Omega_F(V)}$.

Now since $G_F$ is dense in $G = O(V)$, then by continuity of multiplication $\Omega_F(V,Q)\overline{\Omega_F(V)} = \overline{\Omega(V,Q)} = \Omega(V,Q)$, the commutator subgroup of $O(V)$. Hence we have

**Theorem 4.** Let $(V,Q)$ be a semi-simple quadratic space of dimension $\geq 5$ containing isotropic vectors. Then the group $\Omega(V,Q)/Z(V,Q) \cap \Omega(V,Q)$, where $\Omega(V,Q) =$ closure of the group of commutators of $O(V,Q)$, and $Z(V,Q) =$ center of $O(V,Q)$, is a simple topological group.

**Remark 1.** If we provide the general linear group of an infinite dimensional linearly topologized vector space with the finite topology, we get a topological group. Using the same type of definitions and analysis with the aid of the problems in Bourbaki [2, pp. 97–99], we can prove easily a similar theory for the projective special linear group of the general linear group.

**Remark 2.** For the spaces of the type considered in theorem 4 we have a new way to prove that $O(V,Q)$ is not compact; namely, if $O(V,Q)$ is compact, then, since it is also totally disconnected, there exist arbitrarily small open normal subgroups. But this contradicts theorem 3, and hence $O(V,Q)$ could not be compact.

Finally, let us consider when $\Omega(V,Q) = \Omega(V,Q)'$. Since $\overline{\Omega_F(V,Q)} = \Omega(V,Q)$, it suffices to know when $\Omega_F(V,Q)$ is dense in $O(V,Q)$. Again, since $\overline{O(V,Q)} = \overline{G_F}$, it suffices to ask when $\overline{\Omega_F(V,Q)} \supseteq \overline{G_F}$. That is, for any $\sigma \in G_F$, and any $E \in \mathcal{E}_0$ does there exist an $\eta \in \mathcal{E}(E)$ such that $\sigma \cdot \eta \in \Omega_F$. It is clear that such $\eta$ must exist in order that $\Omega(V,Q) = O(V,Q)$; also if
it does exist, then $\eta \in G_F \cap G(E)$. Hence, it has a norm, $\theta_{w}(\eta) = \theta(\eta)$, for some $W \in E_{0}$. Further, we have $\theta(\sigma \eta) = \theta(\tau) = 1$, since $\sigma \eta = \tau \in \Omega_F(V, Q)$; so that $\theta(\sigma) = \theta(\eta)$. Now $\eta \in O(E^*) \cap G_F$ and we may always pick $\eta$ such that $\theta(\sigma) = \theta(\eta)$ if $E^*$ has isotropic vectors (if it does not, we may not be able to find $\eta$ such that $\theta(\eta) = \theta(\sigma)$). Thus we have

**Theorem 5.** Let $(V, Q)$ be a semi-simple quadratic space of dimension $\geq 5$ containing infinitely many linearly independent isotropic vectors. Then we have the diagram

$$\{1_{V}\} \subseteq Z_{V} \cap \Omega(V, Q) \subseteq \Omega(V, Q) = O(V, Q).$$

A group of order 1 or 2. logical group.

**Bibliography**


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