THE OBLIQUITY-TYPE OF A SET OF VECTORS

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We shall classify Euclidean congruence types of \( n \)-tuples of vectors \( \{\beta_1, \ldots, \beta_n\} = \{\beta_i\} \), according to the following scheme. In Euclidean space of \( 2n \) dimensions \( E_{2n} \), suppose there are \( n \) vectors \( \gamma_1, \ldots, \gamma_n \) from an \( n \)-dimensional subspace \( E_n \), and let \( \varphi_1, \ldots, \varphi_n \) be orthonormal basis vectors for an orthogonal complement \( E_{n-1} \) in \( E_{2n} \) of \( E_n \), such that the vectors \( \beta_i \) may be expressed (to within Euclidean congruence of the \( n \)-tuple) in the form

\[
\beta_i = \gamma_i + a_i \varphi_i, \quad i = 1, \ldots, n.
\]

The congruence type of \( \{\beta_i\} \) of course is uniquely determined by the values of the inner products \( (\beta_i, \beta_j) \). Since \( E_n \) is \( n \)-dimensional, for any \( \{\beta_i\} \) a congruent \( n \)-tuple is expressible in the form (1), at least with all \( a_i \)'s equal to zero. For any expression (1), we have \( (\beta_i, \beta_j) = (\gamma_i, \gamma_j) \) for \( i \neq j \), and

\[
||\beta_i||^2 = (\beta_i, \beta_i) = ||\gamma_i||^2 + a_i^2.
\]

Thus if the off-diagonal values of the inner product matrix \( \{(\beta_i, \beta_j)\} \) are realized by any set of vectors \( \{\gamma_i\} \), we may represent the \( n \)-tuple \( \{\beta_i\} \) in the desired form provided that \( ||\beta_i|| \geq ||\gamma_i|| \) for \( i = 1, \ldots, n \). Accordingly we define the (oblivity) type of \( \{\beta_i\} \) as the minimum possible dimension of the linear subspace \( \langle \gamma_1, \ldots, \gamma_n \rangle \) spanned by the set of vectors \( \{\gamma_i\} \), with respect to which a congruent \( n \)-tuple to \( \{\beta_i\} \) can be expressed as \( \{\gamma_i + a_i \varphi_i\} \). Thus of course in case the \( \beta_i \)'s of an \( n \)-tuple are mutually orthogonal, the type is 0.

For any set of vectors \( \{\gamma_i\} \), which is such that the inner products \( (\gamma_i, \gamma_j) \) satisfy

\[
(\gamma_i, \gamma_j) = (\beta_i, \beta_j), \quad i \neq j, \quad i, j = 1, 2, \ldots, n,
\]

we may replace \( \gamma_j \) by another vector, with satisfaction of the same condition (2), to reduce the dimension of the subspace \( \langle \gamma_1, \ldots, \gamma_n \rangle \) by 1, unless \( \gamma_j \) is in the span of the remaining \( \gamma_i \)'s. We state this as a Lemma.

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Lemma. If \( \gamma_n \notin \langle \gamma_1, \ldots, \gamma_{n-1} \rangle \), then without affecting the values of the off-diagonal inner products, we may replace \( \gamma_n \) by a vector \( \gamma_n' \), to reduce the dimension of \( \langle \{ \gamma_i \} \rangle \).

Proof. In case \( \dim \langle \gamma_1, \ldots, \gamma_{n-1} \rangle = 0 \), the type of \( \{ \beta_i \} \) is zero, and we may replace \( \gamma_n \) by a zero vector, increasing the value of \( a_n \) to maintain congruence. Otherwise, let \( \delta_n \) be a unit vector in \( E_n \) which is orthogonal to \( \langle \gamma_1, \ldots, \gamma_{n-1} \rangle \). Then replacing \( \gamma_n \) by any vector of the form \( \gamma_n' = \gamma_n + d_n \delta_n \), the off-diagonal inner products are not affected. We have

\[
(\gamma_n', \gamma_n') = (\gamma_n, \gamma_n) + d_n [2(\gamma_n, \delta_n) + d_n].
\]

The choice \( d_n = -\|\gamma_n\| \cos \theta_n \), where \( \theta_n \) is the angle between \( \gamma_n \) and \( \delta_n \), reduces \( (\gamma_n, \gamma_n) \) to its minimum possible value

\[
(\gamma_n', \gamma_n') = (\gamma_n, \gamma_n)(1 - \cos^2 \theta_n) = (\gamma_n, \gamma_n) \sin^2 \theta_n,
\]

and also subtracts off the component of \( \gamma_n \) in the direction of \( \delta_n \), placing \( \gamma_n' \) in the subspace \( \langle \gamma_1, \ldots, \gamma_{n-1} \rangle \).

Theorem 1. For each \( n \)-tuple \( \{\beta_i\} \), the type exists, and its value is at most \( n-1 \). (For arbitrary non-zero scalars \( b_1, \ldots, b_n \), the type of \( \{b_1 \beta_1, \ldots, b_n \beta_n\} \) is the same as that of \( \{\beta_1, \ldots, \beta_n\} \).)

Proof. Again consider the symmetric matrix of inner products \( \{(\beta_i, \beta_j)\} \). If one of the \( \beta_i \)'s, say \( \beta_n \), is orthogonal to the span of the others, then we may replace \( \beta_n \) by 0 without affecting the values of the off-diagonal inner products. Then if another \( \beta_i \), say \( \beta_{n-1} \), is orthogonal to the span of the remaining \( \beta_i \)'s, it also may be replaced by 0 without affecting the values of the off-diagonal inner products; and so on. An \( n \)-tuple congruent to the original \( \beta_i \)'s may be expressed in the form \( \beta_1, \beta_2, \ldots, \beta_k, a_{k+1} \varphi_{k+1}, \ldots, a_n \varphi_n \). In any case of \( k < n \), we have therefore that type \( \{\beta_i\} \leq k < n \).

In case no \( \beta_i \) is orthogonal to the span of the others, \( k = n \), by the Lemma we have that the type is \( \leq n-1 \). Also in the case \( k < n \) of the preceding paragraph, the type is \( \leq k-1 \). The process indicated in the proof of the Lemma may be continued until we have the situation that each \( \gamma_j \) is in the span of the other \( \gamma_i \)'s. Let us refer to this property of the set of vectors \( \{\gamma_i\} \) as the span property.

Converse Lemma. If a set of vectors \( \{\gamma_i\} \) has the span property, then there does not exist a congruent set of vectors \( \{\gamma_i' + a_i \varphi_i\} \) with \( \dim \langle \{\gamma_i'\} \rangle < \)
dim \langle \{ \gamma_i \} \rangle. With the same hypothesis concerning the set \{ \gamma_i \}, for any set of vectors \{ \gamma'_i + a_i \varphi_i \} such that the correspondence

\[ \gamma'_i \leftrightarrow \gamma'_i + a_i \varphi_i \]

is a congruence, we have that necessarily \( a_i = 0 \) for \( i = 1, \ldots, n \).

**Proof.** If a set \{ \beta_i \} has the span property, then of course any congruent set \{ \gamma'_i \} has the property. If \( a_i \neq 0 \), then \( \gamma'_i + a_j \varphi_j \) cannot be in the span of the other \( (\gamma'_i + a_i \varphi_i)'s \), because \( \varphi_1, \ldots, \varphi_n, E_n \) are mutually orthogonal. No linear combination of the other vectors can cancel the non-zero coefficient \( a_j \).

**Theorem 2.** For a set of vectors \{ \beta_i \} and any two sets of vectors \{ \gamma_i + a_i \varphi_i \} and \{ \gamma'_i + a'_i \varphi_i \}, in which \{ \gamma_i \} and \{ \gamma'_i \} both have the span property, in case the correspondences

\[ \gamma_i + a_i \varphi_i \leftrightarrow \beta_i \quad \text{and} \quad \gamma'_i + a'_i \varphi_i \leftrightarrow \beta_i \]

are congruences, then necessarily \( \gamma_i \leftrightarrow \gamma'_i \) is a congruence, and for \( i = 1, \ldots, n \), we have \( a_i = \pm a'_i \). (Our “congruence” includes the possibility of an involutory isometry, or “mirror image” situation, in which \{ \beta_i' \} could not be brought into coincidence with \{ \beta_i \} by an orthogonal transformation of determinant \( +1 \).)

**Proof.** It follows from our hypothesis that the correspondence \( \gamma_i \leftrightarrow \gamma'_i + (a'_i \pm a_i) \varphi_i \) is a congruence. By the Converse Lemma, for each \( i = 1, \ldots, n \) we must have \( a'_i \pm a_i = 0 \), and therefore that \( \gamma_i \leftrightarrow \gamma'_i \) is a congruence.

**Corollary.** Given a set of vectors \{ \beta_i \}, for any expression of the vectors in the form \( \beta_i = \gamma_i + a_i \varphi_i \), \( i = 1, \ldots, n \), in which the set \{ \gamma_i \} has the span property, we have that the dimension of \( \langle \gamma_1, \ldots, \gamma_n \rangle \) is as small as possible, so that the type of \{ \beta_i \} is equal to that dimension.

**Theorem 3.** In case type \{ \beta_i \} = \text{dim} \langle \gamma_1, \ldots, \gamma_n \rangle = m, the inner product matrix \( \langle (\gamma_i, \gamma_j) \rangle \) (which agrees off the diagonal with \( \langle (\beta_i, \beta_j) \rangle \)) is of the form \( CCT^T \), where \( C \) is an \( n \) by \( m \) matrix, and \( C^T \) is its transposed matrix.

**Proof.** We may choose an orthonormal basis \( \delta_1, \ldots, \delta_n \) for \( E_n \), such that \( \langle \delta_1, \ldots, \delta_m \rangle = \langle \gamma_1, \ldots, \gamma_n \rangle \). Then

\[ \gamma_1 = c_{11} \delta_1 + \ldots + c_{1m} \delta_m, \ldots, \gamma_n = c_{n1} \delta_1 + \ldots + c_{nm} \delta_m; \]

the matrix of coefficients \( C = \{ c_{ij} \} \) is the required matrix.

Representing vectors by their coefficients with respect to an orthonormal basis in \( E_n \), the set of vectors
\[ \gamma_1 = (1,1,1,\ldots,1,0) \]
\[ \gamma_2 = (1,0,0,\ldots,0,0) \]
\[ \gamma_3 = (0,1,0,\ldots,0,0) \]
\[ \ldots \ldots \ldots \ldots \]
\[ \gamma_n = (0,0,0,\ldots,1,0) \]

has the span property, with

\[ \dim \langle \gamma_1, \ldots, \gamma_n \rangle = n - 1. \]

If \( A \) is any linear transformation of rank \( k \) on \( \langle \gamma_1, \ldots, \gamma_n \rangle \), then the set of transforms \( A\gamma_1, \ldots, A\gamma_n \) has the span property. Also it is geometrically obvious that there are sets of any number \( n \) of vectors in the plane, or in a line, which have the span property. Therefore for each integer \( k \) between 0 and \( n - 1 \), inclusive, there exists a linearly independent set \( \{\beta_i\} \) which is of obliquity type \( k \).

The author has in mind application of the obliquity type to classification of convex polytopes. At each vertex of a polytope, consider the set of edge vectors originating at the vertex. In case of an \( n \)-simplex, the type can be zero for at most one vertex. If the type is 0 at one vertex, then necessarily it is 1 at all other vertices. The type of an equilateral \( n \)-simplex is 1 at each vertex. This follows from the fact that the equilateral \( n \)-simplex may be congruently embedded in \( E_{n+1} \) with its vertices at \( (a,0,\ldots,0), \ldots, (0,\ldots,0,a) \). Translation through say \( (-a,0,\ldots,0) \) places one vertex at the origin, and the set of vectors from the origin to the other vertices clearly is of type 1. Similarly, it may be seen that a simplex which is of type 1 at one vertex, must be of type \( \leq 2 \) at all of its other vertices; and that for any Euclidean simplex, if the minimum type among the vertices is \( k \), then each vertex is of type either \( k \) or \( k+1 \).

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