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A GENERALIZATION OF A THEOREM OF FROSTMAN[1]

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1. Introduction.

The theorem to which the title of this paper refers [1, p. 111] concerns inner functions in the open unit disc U; these are the bounded holomorphic functions f in U whose radial limits

$$f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) \tag{1}$$

satisfy the equality $|f^*(e^{i\theta})| = 1$ for almost all real θ. The theorem asserts that *if f is inner, then*

$$\lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \frac{f(re^{i\theta}) - \alpha}{1 - \bar{\alpha} f(re^{i\theta})} \right| d\theta = 0 \tag{2}$$

for nearly all $\alpha \in U$.

Here, and in the rest of this paper, the phrase "nearly all α" means that there is a set E in the plane (the "exceptional set") whose logarithmic capacity is zero, such that the property in question holds for every α in the complement of E.

Frostman's proof applies, almost verbatim, to inner functions in polydiscs. However, a slight reformulation of the original theorem suggests a much more farreaching generalization:

If f is inner and $\alpha \in U$ then $\log|1 - \bar{\alpha} f|$ is a bounded harmonic function in U. Hence

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log|1 - \bar{\alpha} f(re^{i\theta})|\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|1 - \bar{\alpha} f^*(e^{i\theta})|\, d\theta$$

if $0 < r < 1$. Also

$$\log|1 - \bar{\alpha} f^*(e^{i\theta})| = -\log|f^*(e^{i\theta}) - \alpha|$$

if $|f^*(e^{i\theta})| = 1$. So Frostman's theorem is equivalent to the assertion that *if f is inner, then*

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$$\lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{i\theta}) - \alpha| \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f^*(e^{i\theta}) - \alpha| \, d\theta \tag{3}$$

for nearly all α. [For $|\alpha| \geqq 1$, (3) always holds if $|f| < 1$.]

It now seems reasonable to ask whether (3) remains true, say, in the class of all bounded holomorphic functions in U, or, more generally, in H^p-spaces. It turns out that the answer is affirmative, in an even larger subclass of the functions of bounded characteristic, not only in one variable but in several, and that the exceptional sets are always the same, namely sets of logarithmic capacity zero.

Observe that the left side of (3) is the least harmonic majorant of $\log|f-\alpha|$, evaluated at the origin; the right side is the Poisson integral of $\log|f^* - \alpha|$, also evaluated at the origin. Our generalization (Theorem 4) will be stated in terms of these two harmonic functions which are associated with $\log|f-\alpha|$.

2. Definitions.

We fix a positive integer n. The *polydisc* U^n is the cartesian product of n copies of U. In other words, U^n is the set of all $z = (z_1, \ldots, z_n)$ in the space C^n of n complex variables such that $|z_i| < 1$ for $i = 1, \ldots, n$. The *distinguished boundary* of U^n is the torus T^n; $w = (w_1, \ldots, w_n) \in T^n$ provided that $|w_i| = 1$ for $i = 1, \ldots, n$. The Haar measure of T^n will be denoted by dm.

For $z = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) \in U^n$, $w = (e^{it_1}, \ldots, e^{it_n}) \in T^n$, the *Poisson kernel* $P(z,w)$ is the product of the one-variable kernels

$$P(z,w) = \prod_{k=1}^{n} \frac{1 - r_k^2}{1 - 2r_k \cos(\theta_k - t_k) + r_k^2}. \tag{4}$$

If $\varphi \in L^1(T^n)$ we denote its Poisson integral by $P[\varphi]$:

$$P[\varphi](z) = \int_{T^n} P(z,w)\, \varphi(w)\, dm(w), \qquad z \in U^n . \tag{5}$$

More generally, if μ is a measure on T^n, its Poisson integral $P[d\mu]$ is defined as in (5), with $d\mu$ in place of $\varphi\, dm$.

Every Poisson integral is *n-harmonic* in U^n. This means that $P[d\mu]$ is harmonic in each of the variables $z_1, \ldots, z_n$.

A detailed discussion of n-harmonic functions and Poisson integrals is presented in Chapter XVII of [3].

3. The classes N and N_*.

Let $H(U^n)$ be the class of all holomorphic functions in U^n. Associate with each $f \in H(U^n)$ a family of functions f_r on T^n:

$$f_r(w) = f(rw), \qquad 0<r<1, \; w \in T^n . \tag{6}$$

The class N (for Nevanlinna) consists of all $f \in H(U^n)$ such that

$$\sup_{0<r<1} \int \log^+ |f_r| < \infty . \tag{7}$$

[Here and in the sequel the integral extends over T^n, with respect to dm, unless something else is indicated.]

We say that $f \in N_*$ if $f \in H(U^n)$ and if the functions $\log^+ |f_r|$ have uniformly absolutely continuous integrals. Explicitly, what is required is that to each $\varepsilon > 0$ there should exist a $\delta > 0$ such that

$$\int_A \log^+ |f(rw)| \, dm(w) < \varepsilon$$

for all $A \subset T^n$ with $m(A) < \delta$, and for all $r \in (0,1)$. It is clear that $N_* \subset N$.

We observe, in passing, that if Ψ is a positive increasing function on $[0,\infty)$ such that $\Psi(t)/t \to \infty$ as $t \to \infty$, if $f \in H(U^n)$, and if

$$\sup_{0<r<1} \int \Psi(\log^+ |f_r|) < \infty$$

then $f \in N_*$. The proof is as in [3; vol. 1, p. 143]. Taking $\Psi(t) = e^{pt}$ (with $p > 0$) we see that N_* contains all H^p-spaces.

4. Harmonic majorants.

Suppose $f \in H(U^n)$, $f \not\equiv 0$. If $0 < r < 1$, define

$$u_r[f](z) = \int_{T^n} P(r^{-1}z, w) \log |f(rw)| \, dm(w) \tag{8}$$

for those $z = (z_1, \ldots, z_n)$ which have $|z_i| < r$ for $i = 1, \ldots, n$. The fact that $\log |f|$ is subharmonic in each variable implies that $u_r[f](z)$ increases to a limit $u[f](z)$, for each $z \in U^n$, as $r \to 1$, and that [3; vol. II, pp. 321–2]

$$\log |f| \leqq u[f] . \tag{9}$$

If $f \in N$, (7) and (8) show that $u[f](z) < \infty$ for each $z \in U^n$, so $u[f]$ is n-harmonic, by Harnack's theorem.

It is not hard to see that $u[f]$ is actually the *least* n-harmonic majorant of $\log|f|$, in the following sense: If u is n-harmonic and if $\log|f| \leqq u$ in U^n, then actually $u[f] \leqq u$.

We can now state our results.

THEOREM 1. *Suppose* $f \in N$.

(a) *The radial limits*

$$f^*(w) = \lim_{r\to 1} f(rw) \tag{10}$$

exist for almost all $w \in T^n$.

(b) *If* $f \not\equiv 0$ *then* $\log|f^*| \in L^1(T^n)$ *and there is a singular measure* σ_f *on* T^n *such that*

$$u[f] = P[\log|f^*| + d\sigma_f]\,. \tag{11}$$

THEOREM 2. *If* $f \in N_*$, $f \not\equiv 0$, *then*

$$u[f] \leqq P[\log|f^*|]\,. \tag{12}$$

So $\sigma_f \leqq 0$.

THEOREM 3. *If* $f \in N$ *then* $\sigma_{f-\alpha} \geqq 0$ *and*

$$u[f-\alpha] \geqq P[\log|f^* - \alpha|] \tag{13}$$

for nearly all α.

THEOREM 4. *If* $f \in N_*$ *then* $\sigma_{f-\alpha} = 0$ *and*

$$u[f-\alpha] = P[\log|f^* - \alpha|] \tag{14}$$

for nearly all α.

PROOFS. Theorem 1 (a) is of course not new (in fact, much more is known concerning non-tangential limits; see Chapter XVII of [3]) but the following easy reduction to the corresponding one-variable theorem may have some interest. Since $w \to e^{i\theta}w$ is, for each real θ, a measure preserving map of T^n onto T^n we have

$$\int_{T^n} \log^+|f(rw)|\, dm(w) = \int_{T^n} dm(w) \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+|f(re^{i\theta}w)|\, d\theta\,.$$

The integral on the left is bounded as $r \to 1$. Hence the inner integrals on the right are bounded for almost all w, which says that the functions $\lambda \to f(\lambda w)$ are, for almost all $w \in T^n$, of class N in U; for any of these w, $\lim f(re^{i\theta}w)$ exists for almost all θ; hence $f^*(w)$ exist a.e. on T^n.

To prove Theorem 1 (b), define

$$A_r = \int \log^+|f_r|, \qquad B_r = \int \log^-|f_r|, \qquad C_r = \int \log|f_r|\,. \tag{15}$$

Since $\log|f|$ and hence $\log^+|f|$ are subharmonic in each variable, A_r and C_r do not decrease as $r \to 1$; A_r is bounded; $C_r \leqq A_r$; since $f \not\equiv 0$, $C_r > -\infty$; and $B_r = A_r - C_r$. It follows that B_r has a finite limit, as $r \to 1$. By Fatou's lemma,

$$(16) \qquad \int \log^+|f^*| \leqq \lim A_r, \qquad \int \log^-|f^*| \leqq \lim B_r,$$

hence $\log|f^*| \in L^1(T^n)$.

We just saw that $A_r + B_r$ is bounded on $(0,1)$; i.e., the functions $\log|f_r|$ form a bounded family in $L^1(T^n)$. If z is fixed in U^n, then

$$\lim P(r^{-1}z, w) = P(z,w)$$

uniformly for $w \in T^n$, as $r \to 1$. The definition of $u[f]$ therefore shows that

$$(17) \qquad u[f](z) = \lim_{r\to 1} P[\log|f_r|](z), \quad z \in U^n.$$

It follows from the L^1-boundedness of $\{\log|f_r|\}$ that there is a sequence $r_i \to 1$ such that $\log|f_{r_i}|$ converges weakly (in the dual space of $C(T^n)$) to a measure μ on T^n, and hence that $u[f] = P[d\mu]$. To prove (11) we have to show that the Radon–Nikodym derivative of μ is $\log|f^*|$, and this will follow [3; vol. II, p. 313] from the relation

$$(18) \qquad \lim_{r\to 1} u[f](rw) = \log|f^*(w)| \quad \text{a.e. on } T^n.$$

Note that the left side of (18) exists a.e. since $u[f] = P[d\mu]$. Denote it by $u^*(w)$. By (9), $u^* \geqq \log|f^*|$ a.e. Fatou's lemma implies that $\int (u^* - \log|f^*|)$ does not exceed the lower limit of

$$(19) \qquad \int_{T^n} u[f](rw)\, dm(w) - \int_{T^n} \log|f(rw)|\, dm(w).$$

The first integral in (19) is $u[f](0)$, since $u[f]$ is n-harmonic; the second integral is $u_r[f](0)$, as in (8). Hence the limit of (19), as $r \to 1$, is 0. So $u^* \leqq \log|f^*|$ a.e. This proves (18), and Theorem 1 is complete.

Now suppose $f \in N_*$, $f \not\equiv 0$. Since $\log^+|f_r| \to \log^+|f^*|$ a.e., the uniform absolute continuity of the integrals of $\log^+|f_r|$ implies that

$$(20) \qquad P[\log^+|f^*|] = \lim_{r\to 1} P[\log^+|f_r|].$$

By Fatou's lemma,

$$(21) \qquad P[\log^-|f^*|] \leqq \liminf_{r\to 1} P[\log^-|f_r|].$$

If we subtract (21) from (20), (17) gives Theorem 2.

Before turning to the proof of Theorem 3 let us observe that the obvious inequality

$$\log^+|f-\alpha| \leqq |\alpha| + \log^+|f| \tag{22}$$

implies that $f-\alpha \in N$ if $f \in N$ and that $f-\alpha \in N_*$ if $f \in N_*$.

Our first aim in the proof of Theorem 3 will be to show that

$$\lim_{r\to 1}\int \log^-|f_r-\alpha| = \int \log^-|f^*-\alpha| \tag{23}$$

for nearly all α.

To do this it suffices to show that every compact set K of *positive* logarithmic capacity contains points α for which (23) holds. So let K be such a set. There is a positive measure μ on K such that the potential

$$G(\lambda) = \int_K \log^-|\lambda-\alpha|\, d\mu(\alpha) \tag{24}$$

is continuous in the whole plane [2; p. 84]; having compact support, G is also bounded. (In Frostman's proof it was enough to know the existence of a μ for which G was bounded; our present setting requires the continuity of G.) Put

$$B_r(\alpha) = \int_{T^n} \log^-|f(rw)-\alpha|\, dm(w)\,. \tag{25}$$

Then

$$\int_K B_r(\alpha)\, d\mu(\alpha) = \int_{T^n} G(f(rw))\, dm(w)\,. \tag{26}$$

We know, by the reasoning following (15), that $\lim B_r(\alpha)$ exists. Call it $B(\alpha)$. Apply Fatou's lemma to the left side of (26) and the dominated convergence theorem to the right; this depends on the continuity and boundedness of G. We obtain

$$\begin{aligned}\int_K B(\alpha)\, d\mu(\alpha) &\leqq \liminf_{r\to 1}\int_K B_r(\alpha)\, d\mu(\alpha)\\ &= \liminf_{r\to 1}\int_{T^n} G(f(rw))\, dm(w)\\ &= \int_{T^n} G(f^*(w))\, dm(w)\\ &= \int_K d\mu(\alpha)\int_{T^n}\log^-|f^*(w)-\alpha|\, dm(w)\,.\end{aligned} \tag{27}$$

But Fatou's lemma implies also that

$$\int_{T^n} \log^- |f^*(w) - \alpha| \, dm(w) \leqq B(\alpha) \tag{28}$$

for every α. We conclude from (27) and (28) that (23) holds for almost all (relative to μ) $\alpha \in K$. Thus (23) holds for nearly all α.

Next we claim that actually

$$\lim_{r \to 1} \int \Big| \log^- |f_r - \alpha| - \log^- |f^* - \alpha| \Big| = 0 \tag{29}$$

for every α for which (23) holds. This is a consequence of the following lemma:

LEMMA. *If $\{\varphi_i\}$ is a sequence of measurable functions on a measure space X, if $\varphi_i \to \varphi$ pointwise* a.e., *and if*

$$\lim \int_X |\varphi_i| = \int_X |\varphi| < \infty ,$$

then

$$\lim \int_X |\varphi_i - \varphi| = 0 .$$

This may not be as well known as it deserves to be, so we include a proof. Assume $\int_X |\varphi_i| = 1$, without loss of generality. If $\varepsilon > 0$, X can be partitioned into sets A, B such that $\varphi_i \to \varphi$ uniformly on A, A has finite measure, and $\int_B |\varphi| < \varepsilon$ (Egoroff). For large i,

$$\int_B |\varphi_i| = 1 - \int_A |\varphi_i| < 1 + \varepsilon - \int_A |\varphi| = \varepsilon + \int_B |\varphi| < 2\varepsilon$$

so that

$$\int_X |\varphi_i - \varphi| < 3\varepsilon + \int_A |\varphi_i - \varphi| ,$$

which proves the lemma.

To apply the lemma to our situation, let $r_i \to 1$ and put $\varphi_i = \log^- |f_{r_i} - \alpha|$, $\varphi = \log |f^* - \alpha|$. This gives (29).

It follows that

$$P[\log^- |f^* - \alpha|] = \lim_{r \to 1} P[\log^- |f_r - \alpha|] \tag{30}$$

if (23) holds, whereas Fatou's lemma gives

$$P[\log^+ |f^* - \alpha|] \leqq \lim_{r \to 1} P[\log^+ |f_r - \alpha|] \tag{31}$$

for every α. If we subtract (30) from (31) we obtain (13) and hence Theorem 3.

Finally, Theorem 4 is an immediate corollary of Theorems 2 and 3.

5. Examples.

Strict inequality can hold in (13). In fact, if

$$f(z) = \exp\left\{\frac{1+z}{1-z}\right\}, \qquad z \in U, \tag{32}$$

then

$$u[f-\alpha](0) = 1 + P[\log|f^*-\alpha|](0) \tag{33}$$

for *every* complex number α. We omit the proof of this. It can also happen, for certain α, that (13) holds for some z but not for all. An example is

$$f(z) = \exp\left\{\frac{z}{1-z^2}\right\}, \qquad z \in U, \tag{34}$$

where $P[\log|f^*|](z)=0$ for all z but $u[f](z)$ assumes both positive and negative values in U. That (14) may fail for all α in a preassigned compact set of capacity zero was already noted by Frostman in [1].

6. Further generalizations.

Let B^n be the unit ball in C^n: $z=(z_1,\ldots,z_n) \in B^n$ provided that $|z_1|^2+\ldots+|z_n|^2<1$. Everything said in this paper holds with B^n in place of U^n if the following changes are made: T^n must be replaced by the sphere defined by $\sum|z_i|^2=1$, the Haar measure on T^n must be replaced by the normalized rotation invariant measure on this sphere, $P(z,w)$ must be replaced by the Poisson kernel for the ball, and "n-harmonic" is to be replaced by "harmonic". We leave it to the reader to verify that theorems and proofs are unaffected by these changes.

There are undoubtedly other types of regions in which similar results hold.

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