THE LATTICE-GRAPH OF
THE TOPOLOGY OF A TRANSITIVE DIRECTED GRAPH

SABRA SULLIVAN ANDERSON and GARY CHARTRAND

A directed graph $D$ is transitive if whenever the directed lines $uv$ and $vw$ are in $D$, then the directed line $uw$ is also in $D$. It was shown by Evans, Harary, and Lynn [1] that there is a one-to-one correspondence between the labeled topologies on $n$ points and the labeled transitive directed graphs with $n$ points; therefore, under this correspondence, there is associated a topology with every labeled transitive directed graph $D$. We call the directed graph of the lattice of open sets determined by this topology the lattice-graph of the topology of $D$. It is the object of this note to investigate the lattice-graphs of the topologies of transitive directed graphs. In particular, we characterize those transitive directed graphs whose topologies have isomorphic lattice-graphs.

Graphical definitions not given here may be found in [2].

The associated topology of a transitive directed graph.

We find it convenient to describe here the one-to-one correspondence between the labeled topologies with $n$ points and the labeled transitive directed graphs on $n$ points, as it appeared in [1].

With each topology $\mathcal{O}$ on a set $V$ of $n$ points, a directed graph $D(\mathcal{O})$ can be defined on $V$ by drawing a line directed from $u \in V$ to $v \in V$ if and only if $u$ is in every open set containing $v$. It is then easy to verify that $D(\mathcal{O})$ is transitive and uniquely determined by $\mathcal{O}$. For example, the topology $\mathcal{O}$ on $V = \{a, b, c, d\}$, defined by

$$\mathcal{O} = \{\emptyset, \{a\}, \{a, b\}, V\},$$

gives rise to the directed graph $D_1 = D_1(\mathcal{O})$ in Figure 1.

Conversely, with each labeled transitive directed graph $D$ having $n$ points, a topology $\mathcal{O} = \mathcal{O}(D)$ on this set $V$ of points is induced by the basis $B$ whose elements consist of the sets:

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\[ Q(u) = \{u\} \cup \{v \in V \mid vu \text{ is a directed line in } D\} \]

for all \( u \in V \). The transitive directed graph \( D_2 \) of Figure 2 generates the basis
\[ B = \{Q(a) = \{a\}, Q(b) = \{b\}, Q(c) = \{c, b\}, Q(d) = V, Q(e) = \{e, a\}\} \]

which induces the topology
\[ \mathcal{O}(D_2) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, e\}, \{a, b, c\}, \{a, b, e\}, \{a, b, c, e\}, V\} . \]

The lattice-graph associated with a topology.

With each topology \( \mathcal{O} \) on a finite set, a lattice, whose directed graph is called the lattice-graph of \( \mathcal{O} \), may be associated in a natural way by letting the open sets \( O_i \) be represented by points \( v_i \) and drawing an arc from \( v_i \) to \( v_j \) if \( O_i \subset O_j \) and there exists no \( O_h \) such that \( O_i \subset O_h \subset O_j \). The lattice-graph of \( \mathcal{O}(D_2) \) is shown in Figure 3.
As we have seen, given a labeled transitive directed graph \( D \), there corresponds a topology \( \mathcal{O}(D) \) on the point set of \( D \) and thus a lattice-graph \( \mathcal{L} \) associated with the open sets of \( \mathcal{O} \). We refer to \( \mathcal{L} = \mathcal{L}(D) \) as the lattice-graph of the topology of \( D \). We now investigate the relationship between transitive directed graphs and the lattice-graphs of their topologies. In order to describe this relationship, a few intermediate concepts are useful.

A clique in a directed graph is a maximal complete symmetric subgraph. Although the cliques of a directed graph \( D \) do not, in general, partition the point set of \( D \), such is the case if \( D \) is transitive.

**Lemma 1.** The cliques of a transitive directed graph partition its point set.

**Proof.** Clearly, every point of a transitive directed graph \( D \) lies in at least one clique. Suppose \( v \) is a point of \( D \) which lies in two distinct cliques \( C_1 \) and \( C_2 \). Then if \( u \in C_1 \) and \( w \in C_2 \), both the lines \( uw \) and \( wu \) are in \( D \) since \( D \) is transitive. However, this implies that \( C_1 \) and \( C_2 \) are properly contained in a complete symmetric subgraph contradicting the fact that \( C_1 \) and \( C_2 \) are maximal. Hence \( v \) belongs to a single clique and the cliques partition the point set of \( D \).

In view of this result, we define the clique-condensation graph \( D' \) of a transitive directed graph \( D \) to be that directed graph whose points are in one-to-one correspondence with the cliques \( \{C_1, C_2, \ldots\} \) of \( D \) and in which there is a line directed from the point corresponding to \( C_i \) to the point corresponding to \( C_j \) in \( D' \) if there is a line directed from a point of \( C_i \) to a point of \( C_j \) in \( D \). A transitive directed graph \( D_3 \) and its clique-condensation graph are shown in Figure 4.

**Theorem 1.** The clique-condensation graph \( D' \) of a transitive directed graph \( D \) is transitive and asymmetric.

**Fig. 4.**
Proof. By the transitivity of $D$, if there is a line directed from a point of $C_i$ to a point of $C_j$, then there must be a line from every point of $C_i$ to every point of $C_j$. Furthermore, there can be no lines directed from $C_j$ to $C_i$ since cliques are maximal.

Corollary 1a. A transitive directed graph $D$ is isomorphic to its clique-condensation graph if and only if $D$ is asymmetric.

It is not difficult to find two different transitive directed graphs whose topologies have isomorphic lattice-graphs, but, as we shall see, such directed graphs must enjoy a common property.

Lemma 2. The topologies of a transitive directed graph $D$ and its clique-condensation graph $D'$ have isomorphic lattice-graphs.

Proof. If a point $v$ of $D$ lies in an open set $O$ of the topology $\mathcal{O}(D)$, then every point in the clique to which $v$ belongs also lies in $O$. Since the transitivity relation among the cliques in $D$ is the same as that among the corresponding points of $D'$, $\mathcal{L}(D)$ and $\mathcal{L}(D')$ are isomorphic, where the point

$$\{C_{i_1}, C_{i_2}, \ldots, C_{i_k}\}$$

of $\mathcal{L}(D')$ corresponds to the point

$$C_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_k}$$

of $\mathcal{L}(D)$.

Theorem 2. The lattice-graphs $\mathcal{L}(D_1)$ and $\mathcal{L}(D_2)$ of the topologies of two transitive directed graphs $D_1$ and $D_2$ are isomorphic if and only if the clique-condensation graphs $D_1'$ and $D_2'$ are isomorphic.

Proof. Suppose $D_1'$ and $D_2'$ are isomorphic. Then clearly $\mathcal{L}(D_1')$ and $\mathcal{L}(D_2')$ are isomorphic. However, by Lemma 2, $\mathcal{L}(D_1)$ and $\mathcal{L}(D_1')$ are isomorphic as are $\mathcal{L}(D_2)$ and $\mathcal{L}(D_2')$. Hence $\mathcal{L}(D_1)$ is isomorphic to $\mathcal{L}(D_2)$.

Conversely, to show $\mathcal{L}(D_1)$ and $\mathcal{L}(D_2)$ isomorphic implies $D_1'$ and $D_2'$ isomorphic, it is equivalent to show that the lattice-graph $\mathcal{L}(D)$ uniquely determines the topology $\mathcal{O}(D')$, or that $\mathcal{L}(D')$ uniquely determines the topology of $D'$, which in turn uniquely determines $D'$, since by Lemma 2, $\mathcal{L}(D)$ and $\mathcal{L}(D')$ are isomorphic. Now the points of $\mathcal{L}(D')$ correspond to open sets of a topology on some finite set $V$. If $uv$ is a directed line of $\mathcal{L}(D')$, then $u$ and $v$ correspond to open sets $O_1$ and $O_2$, respectively, for which $O_1 \subset O_2$ but no open set $O_3$ exists having $O_1 \subset O_3 \subset O_2$. The set $O_2 \setminus O_1$ necessarily consists of a single element of $V$, for if $\{a, b\} \subset O_2 \setminus O_1$, then whenever one of $a$ and $b$ is contained in an open set, then so must
the other be contained in the same open set. This implies that both
directed lines ab and ba are in D' contradicting the asymmetry condition
(see Theorem 1). Also, for every point a ∈ V, the smallest open set
containing a corresponds to a point w₁ of L(D') having indegree one,
i.e., having one line directed towards it, for if w₂w₁ and w₃w₁ were directed
lines of L(D'), then the union of the open sets corresponding to w₂ and
w₃ would be a larger open set not containing a, implying that the lines
w₂w₁ and w₃w₁ do not exist. Hence, up to a permutation of V, the one-
to-one correspondence between the points of L(D') and the open sets
defined on V is given as follows. First, identify with each point u of
L(D') having indegree one a distinct point a of V (which is possible by
the preceding discussion). Then for any point v of L(D') associate the
subset of V consisting of the union of any point identified with v and
all points a identified with points u for which there is a directed path
from u to v in L(D'). The correspondence is completed by associating
the empty set ∅ with the remaining unlabeled point of L(D'). This
uniquely determines D' (up to labeling).

Corollary 2a. For any lattice-graph L of the topology of a transitive
directed graph, there exists a unique asymmetric transitive directed graph D
the lattice-graph of whose topology is isomorphic to L.

It therefore follows that every transitive directed graph whose topology
has lattice-graph L can be obtained from the asymmetric directed graph
D whose topology has lattice-graph L by a suitable replacement of the
points of D by complete symmetric directed graphs. Thus, all transitive
directed graphs whose topologies have lattice-graphs L are completely
determined and constructible.

REFERENCES

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EASTERN MICHIGAN UNIVERSITY YPSILANTI, MICH., U.S.A.
THE UNIVERSITY OF MICHIGAN ANN ARBOR, MICH., U.S.A.
WESTERN MICHIGAN UNIVERSITY KALAMZOO, MICH., U.S.A.