A CLOSURE PROBLEM FOR SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS

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1. Introduction.

Let \( \{m_v\}_{v=0}^n \) be a sequence of positive numbers, \( m_0 = 1 \), and let \( C_n \) be the space of complex-valued, \( n \) times differentiable functions \( f \), defined for \( x \geq 0 \), for which

\[
||f||_{(n)}^2 = \sum_{r=0}^n m_r^{-2} \int_0^\infty |f^{(r)}(x)|^2 \, dx < \infty.
\]

Further, let

\[
\sum_{r=0}^n m_r^{-2}z^{2r} = \prod_{j=1}^n (1 + r_j^{-2}z^2), \quad \text{Re} r_j > 0,
\]

and suppose that \( r_i \neq r_j \) for \( i \neq j \).

Theorem 1. Let \( \varphi \in C_n \). Then the extremal problem

\[
\min_{b_j} ||\varphi(x) - \sum_{j=1}^n b_j e^{-r_j x}||_{(n)}
\]

is solved by

\[
\varphi_n(x) = \sum_{j=1}^n b_j^{(n)} e^{-r_j x},
\]

where

\[
\varphi_n^{(v)}(0) = \varphi^{(v)}(0), \quad v = 0, 1, \ldots, n-1.
\]

Proof. Define in \( C_n \) the inner product

\[
(f, g) = \sum_{r=0}^n m_r^{-2} \int_0^\infty f^{(r)}(x) g^{(r)}(x) \, dx.
\]

It is well known that the extremal function \( \varphi_n \) is the unique solution of the system of equations

\[
(1.3) \quad (\varphi - \varphi_n, e^{-r_j x}) = 0, \quad j = 1, 2, \ldots, n.
\]
If \( f \) is an arbitrary function in \( C_n \), partial integrations give for \( 1 \leq \nu \leq n \)

\[
\int_0^\infty f^{(\nu)}(x) e^{-\bar{r}_j x} \, dx = - \sum_{k=1}^\nu \bar{r}_j^{k-1} f^{(\nu-k)}(0) + \bar{r}_j^\nu \int_0^\infty f(x) e^{-\bar{r}_j x} \, dx.
\]

Hence

\[
(f, e^{-\bar{r}_j x}) = \sum_{\nu=0}^n (-1)^\nu m_\nu^{-2} \bar{r}_j^\nu \int_0^\infty f^{(\nu)}(x) e^{-\bar{r}_j x} \, dx
\]

\[
= \sum_{\nu=0}^n (-1)^{\nu-1} m_\nu^{-2} \bar{r}_j^{\nu-1} \int_0^\infty f^{(\nu)}(x) e^{-\bar{r}_j x} \, dx + \sum_{\nu=0}^n (-1)^\nu m_\nu^{-2} \bar{r}_j^{2\nu} \int_0^\infty f(x) e^{-\bar{r}_j x} \, dx
\]

\[
= \sum_{\nu=0}^{n-1} f^{(\nu)}(0) \sum_{\nu=0}^\mu (-1)^\nu m_\nu^{-2} \bar{r}_j^{2\nu-\mu-1}
\]

by (1.2). Thus (1.3) is equivalent to

\[
\sum_{\nu=0}^{n-1} (q^{(\nu)}(0) - q_n^{(\nu)}(0)) \sum_{\nu=0}^\mu (-1)^\nu m_\nu^{-2} \bar{r}_j^{2\nu-\mu-1} = 0, \quad j = 1, 2, \ldots, n.
\]

Obviously, this system has the solution

\[
q_n^{(\nu)}(0) = q^{(\nu)}(0), \quad \mu = 0, 1, \ldots, n - 1.
\]

Since there is a one-to-one correspondence between \( q_n \) and \( \{q_n^{(\nu)}(0)\}_{\mu=0}^{n-1} \), the proof is complete.

On the other hand, the function \( q_n \) has the property

\[
\|q_n\|_{(n)} = \min \|f\|_{(n)}
\]

for all \( f \) in \( C_n \) satisfying

\[
f^{(\nu)}(0) = q^{(\nu)}(0), \quad \mu = 0, 1, \ldots, n - 1
\]

(see [5, pp. 127–128]). Thus, the function in the span of \( \{e^{-\bar{r}_j x}\}_{j=1}^n \) which is the best approximation of \( \varphi \) in the norm (1.1) at the same time solves the problem of interpolating given values (1.5) by a function \( f \) in \( C_n \) with minimal norm.

Now let \( A = \{A_\nu\}_0^\infty \) be a sequence of positive numbers satisfying

\[
A_0 = 1,
\]

\[
\log A_\nu \text{ is a convex function of } \nu,
\]

and

\[
\lim_{\nu \to \infty} (A_\nu/\nu!)^{1/\nu} > 0.
\]
Let $\mathcal{F}_A$ be the Banach space of all complex-valued, infinitely differentiable functions $f(x)$, defined for $x \geq 0$, for which

$$
\|f\|^2 = \sum_{r=0}^{\infty} A_r^{-2} \int_0^{\infty} |f^{(r)}(x)|^2 \, dx < \infty
$$

and suppose that

$$
\sum_{r=0}^{\infty} A_r^{-2} z^{2r} = \prod_{j=1}^{\infty} (1 + r_j^{-2} z^2), \quad r_j > 0,
$$

where $r_i < r_j$ for $i < j$. If $f \in \mathcal{F}_A$ it follows that

$$
\sup_{x \geq 0} |f^{(r)}(x)| \leq K (A_r A_{r+1}), \quad r = 0, 1, 2, \ldots,
$$

where $K$ is a constant (depending on $f$) (see [6, Lemma 6]).

Let $\varphi$ be a fixed function in $\mathcal{F}_A$. It follows that $\varphi \in C_n$ for every $n$ and, by (1.4),

$$
\|\varphi_n\|_{(n)} \leq \|\varphi\|_{(n)} \leq \|\varphi\|.
$$

Using (1.11) it is easy to prove that there exists a subsequence $\{\varphi_{n_k}(x)\}$ converging to an infinitely differentiable function $\psi(x)$ in such a way that

$$
\varphi_{n_k}^{(r)}(x) \to \psi^{(r)}(x), \quad r = 0, 1, 2, \ldots,
$$

uniformly on every interval $[0, a]$ and $\psi(x)$ satisfies (1.10) (see Mandelbrojt [3, pp. 104–105]). But

$$
\psi^{(r)}(0) = \varphi^{(r)}(0), \quad r = 0, 1, 2, \ldots.
$$

If the sequence $A$ has the property that the inequalities (1.10) define a quasi-analytic class, and this is the case if and only if (see Mandelbrojt [3])

$$
\int_0^{\infty} x^{-2} \log (\sum_0^{\infty} x^{2r}/A_r^2) \, dx = \infty,
$$

then, since $\varphi$ and $\psi$ both satisfy (1.10), we have $\psi \equiv \varphi$. This also implies that

$$
\lim_{k \to \infty} \|\varphi - \varphi_{n_k}\|_{(n_k)} = 0.
$$

In the special case $\varphi(x) = e^{-ax}$, $\text{Re} \alpha > 0$, a calculation yields

$$
\|\varphi - \varphi_n\|^2_{(n)} = \frac{1}{2 \, \text{Re} \alpha} \prod_{j=1}^{n} |1 - \alpha/r_j|^2.
$$

This tends to zero as $n \to \infty$ if and only if
\[ \sum_{j=1}^{n} r_j^{-1} = \pi^{-1} \int_{0}^{\infty} x^{-2} \log \left( \sum_{r=0}^{\infty} x^{2r} / m_r^2 \right) dx \rightarrow \infty \]

(for the equality, see [5, p. 130]), that is, if and only if (1.12) holds.

In view of the above-mentioned facts it is natural to consider the following closure problem. Let \( A = \{ A_r \}_0^\infty \) be a sequence of positive numbers satisfying (1.6)–(1.8). Let, for a fixed \( k \geq 1 \), \( \mathcal{F}_{A(k)} \) be the Banach space of all complex-valued, infinitely differentiable functions \( f(x) \), defined for \( x \geq 0 \), for which

\[ (1.13) \quad \| f \|_{k}^2 = \sum_{r=0}^{\infty} k^{-2r} A^2_r \int_{0}^{\infty} |f^{(r)}(x)|^2 dx < \infty, \]

and let \( \mathcal{L}_{A(k)} \) be the normed linear subspace of \( \mathcal{F}_{A(k)} \) which consists of all \( f \) satisfying

\[ (1.14) \quad \| f \|_{1} < \infty. \]

Further, let \( \lambda = \{ \lambda_j \}_1^\infty \) be an increasing sequence of positive numbers tending to infinity and such that

\[ (1.15) \quad \sum_{j=1}^{\infty} \lambda_j^{-1} = \infty. \]

Then our problem is to decide whether or not the set

\[ A = \{ e^{-\lambda_j x} : j = 1, 2, 3, \ldots \} \]

is fundamental in \( \mathcal{L}_{A(k)} \), that is, the span of \( A \) is dense in \( \mathcal{L}_{A(k)} \). (For a similar problem, see Korenbljum [2].)

Of course, the answer of this question will depend on the sequences \( A \) and \( \lambda \), and on \( k \). Concerning \( A \) we speak of the quasi-analytic case if (1.12) holds and the non-quasi-analytic case if

\[ (1.16) \quad \int_{0}^{\infty} x^{-2} \log \left( \sum_{0}^{\infty} x^{2r} / A_r^2 \right) dx < \infty. \]

The subject of this paper was suggested by Professor Lennart Carleson and I wish to express my deep gratitude to him for his valuable and generously given advice.

2. Approximation of \( e^{-ax} \), Re \( a > 0 \).

We start by considering the approximation in \( \mathcal{F}_{A(1)} \) of \( e^{-ax} \), Re \( a > 0 \), by finite linear combinations of functions of the set \( A \).
THEOREM 2. Let
\[ S(R) = \sum_{\lambda_j \in R} \lambda_j^{-1} \]
and
\[ I(R) = \pi^{-1} \int_{1}^{R} y^{-2} \log \left( \sum_{0}^{\infty} y^j/A_j \right) dy. \]

Then, a sufficient condition that every function \( e^{-ax}, \text{Re} \alpha > 0 \), can be approximated arbitrarily closely in \( \mathcal{F}_{A_\omega} \) by finite linear combinations of functions of \( A \) is that
\[ \lim_{R \to \infty} \frac{I(R)}{S(R)} < 1. \]

COROLLARY. In the non-quasi-analytic case every function \( e^{-ax}, \text{Re} \alpha > 0 \), can be approximated arbitrarily closely in \( \mathcal{F}_{A_\omega} \) by finite linear combinations of functions of \( A \).

PROOF OF THE COROLLARY. This is immediate since (1.16) implies that \( I(R) \) is bounded.

PROOF OF THEOREM 2. As is well known, it is sufficient to show that for every bounded, linear functional \( L \) defined on \( \mathcal{F}_{A_\omega} \) it holds true that
\[ L(e^{-j\alpha x}) = 0, \quad j = 1, 2, 3, \ldots, \]
implies
\[ L(e^{-\alpha x}) \equiv 0, \quad \text{Re} \alpha > 0. \]

For arbitrary \( f \) and \( g \) in \( \mathcal{F}_{A_\omega} \) let
\[ (f, g) = \sum_{\nu = 0}^{\infty} A_\nu^{-2} \int_{0}^{\infty} f^{(\nu)}(x) g^{(\nu)}(x) \, dx; \]
this makes \( \mathcal{F}_{A_\omega} \) into a complete inner product space. Then, by the theorem of Fréchet–Riesz, we can represent \( L \) in the form
\[ L(f) = (f, \varphi), \quad \varphi \in \mathcal{F}_{A_\omega}. \]

Let
\[ L(e^{-\alpha x}) \sum_{\nu = 0}^{\infty} (-\alpha)^{\nu} A_\nu^{-2} \int_{0}^{\infty} e^{-\alpha x} \varphi^{(\nu)}(x) \, dx = F(\alpha), \quad \text{Re} \alpha > 0. \]

We observe that \( F(\alpha) \) is holomorphic in the half plane \( \text{Re} \alpha > 0 \), and we have to show that
\[ F(\lambda_j) = 0, \quad j = 1, 2, 3, \ldots, \]
implies
\[ F(\alpha) \equiv 0, \quad \text{Re} \alpha > 0. \]

To do this we use Carleman’s theorem: If \( f(\alpha) \) is holomorphic for \( \text{Re} \alpha \geq 0 \) and if \( r_j e^{i\theta_j}, j = 1, 2, 3, \ldots, \) are the zeros of \( f(\alpha) \) in this half plane, then as \( R \to \infty \)

\[
(2.7) \quad \sum_{r_j \leq R} (r_j^{-1} - r_j R^{-2}) \cos \theta_j = (\pi R)^{-1} \int_{-\frac{1}{4} \pi}^{\frac{1}{4} \pi} \log|f(R e^{i\theta})| \cos \theta \, d\theta + \int_{-\frac{1}{4} \pi}^{\frac{1}{4} \pi} \int_{1}^{R} (y^{-2} - R^{-2}) \log|f(iy)f(-iy)| \, dy + O(1). 
\]

We take \( f(\alpha) = F(\alpha + \delta) \) for a fixed \( \delta > 0 \). Since for \( \text{Re} \alpha \geq 0 \)

\[
\left| \int_{0}^{\infty} e^{-(\alpha+\delta)x} \varphi^{(r)}(x) \, dx \right| \leq (2\delta)^{-1} ||\varphi||_1 A_r,
\]

we have for \( |\alpha| \leq R, \text{Re} \alpha \geq 0 \)

\[
|F(\alpha + \delta)| \leq (2\delta)^{-1} ||\varphi||_1 \sum_{0}^{\infty} (R + \delta)^r / A_r \leq (2\delta)^{-1} ||\varphi||_1 e^{\rho(R+\delta)}
\]

by (1.8), for some constant \( \rho > 0 \). Hence the first term in the right hand member of (2.7) is bounded above as \( R \to \infty \). Further,

\[
(2\pi)^{-1} \int_{1}^{R} (y^{-2} - R^{-2}) \log|F(iy + \delta)F(-iy + \delta)| \, dy
\]

\[
\leq \pi^{-1} \int_{1}^{R} (y^{-2} - R^{-2}) \log((2\delta)^{-1} ||\varphi||_1 \sum_{0}^{\infty} (y + \delta)^r / A_r) \, dy
\]

\[
\leq \pi^{-1} \int_{1}^{R} y^{-2} \log \left( \sum_{0}^{\infty} (y + \delta)^r / A_r \right) \, dy + O(1)
\]

\[
= (1 - \delta)^{-2} I(R) + O(1)
\]

and, for \( 0 < \beta < 1 \),

\[
\sum_{\lambda_j \leq R} ((\lambda_j - \delta)^{-1} - (\lambda_j - \delta) R^{-2}) \geq (1 - \beta^2) S(\beta R). 
\]

Then, by (2.7) and since \( I(R) = I(\beta R) + O(1) \),

\[
(1 - \beta^2) S(\beta R) \leq (1 - \delta)^{-2} I(\beta R) + O(1).
\]

Here \( S(\beta R) \to \infty \) as \( R \to \infty \) by (1.15), and choosing \( \beta \) and \( \delta \) sufficiently
small this leads to a contradiction if (2.3) holds. Hence $F(x + \delta) \equiv 0$ for $\Re x \geq 0$ and thus, since we can take $\delta > 0$ arbitrarily small, $F(x) \equiv 0$ for $\Re x > 0$. This completes the proof of Theorem 2.

3. The non-quasi-analytic case.

In this section we consider the closure problem in the non-quasi-analytic case (1.16). We prove the following theorem which can be considered as an extension of the well-known theorem of Müntz (see Schwartz [4]).

**Theorem 3.** In the non-quasi-analytic case, $A$ is fundamental in $\mathcal{L}_A(\nu)$, $k > 1$.

To prove this theorem we need some lemmas. For a function $f$, defined in a set containing $[0, \infty)$, we always denote its restriction to $[0, \infty)$ by the same symbol $f$.

**Lemma 1.** Let $f \in \mathcal{F}_A(\nu)$, and define, for $\eta > 0$,

$$f_\eta(x) = f(x + \eta), \quad x \geq -\eta.$$

Then

$$\lim_{\eta \to 0^+} \|f - f_\eta\|_k = 0, \quad k \geq 1.$$

**Proof.** Obviously

$$\sum_{v=N}^{\infty} k^{-2v} A_v^{-2} \int_0^\infty |f^{(v)}(x) - f^{(v)}(x + \eta)|^2 \, dx$$

becomes arbitrarily small for $N$ sufficiently large, independent of $\eta$. Having fixed $N$, we can make

$$\sum_{v=0}^{N-1} k^{-2v} A_v^{-2} \int_0^\infty |f^{(v)}(x) - f^{(v)}(x + \eta)|^2 \, dx$$

arbitrarily small by choosing $\eta$ sufficiently small, since

$$\lim_{\eta \to 0^+} \int_0^\infty |f^{(v)}(x) - f^{(v)}(x + \eta)|^2 \, dx = 0.$$

**Lemma 2.** If (1.16) holds, there exists for arbitrary constants $\eta > 0$ and $k_0 > 0$ an infinitely differentiable function $\varphi(x)$, defined on $(-\infty, \infty)$, such that
\( \varphi(x) = \begin{cases} 
0 & \text{for } x \leq 0 \\
1 & \text{for } x \geq \eta 
\end{cases} \)

and for some constant \( C \)

\( \sup_x |\varphi^{(v)}(x)| \leq C k_0^v A_\nu, \quad v = 0, 1, 2, \ldots. \)

**Proof.** By (1.7), the sequence \( \{A_{\nu-1}/A_\nu\}_{1}^{\infty} \) is decreasing. Further, (1.16) is equivalent to (see Mandelbrojt [3])

\( \sum_{1}^{\infty} A_{\nu-1}/A_\nu < \infty. \)

Then we can construct a sequence \( \{B_\nu\}_{0}^{\infty} \) such that

\( B_{\nu-1}/B_\nu = q_\nu A_{\nu-1}/A_\nu, \quad \nu = 1, 2, 3, \ldots, \)

where \( q_\nu \to \infty \) as \( \nu \to \infty \), the sequence \( \{B_{\nu-1}/B_\nu\}_{1}^{\infty} \) is decreasing and

\( \sum_{1}^{\infty} B_{\nu-1}/B_\nu < \infty. \)

It is well known that there exists an infinitely differentiable function \( \varphi(x) \), defined on \( (-\infty, \infty) \), satisfying (3.1) and, for constants \( C_\eta \) and \( K_\eta \) depending on \( \eta \),

\( \sup_x |\varphi^{(v)}(x)| \leq C_\eta K_\eta^v B_\nu, \quad v = 0, 1, 2, \ldots. \)

Hence, by (3.3), the inequalities (3.2) are fulfilled if \( C \) is sufficiently large.

**Lemma 3.** If (1.16) holds and if \( f \in \mathcal{L}_{A(x)} \), \( k > 1 \), then for every \( \varepsilon > 0 \) there exist constants \( \eta > 0 \) (small) and \( L > 0 \) (large) and an infinitely differentiable function \( g(x) \), defined on \( (-\infty, \infty) \), such that \( g(x) \equiv 0 \) outside \( (-\eta, L) \),

\( \sum_{v=0}^{\infty} k^{-2v} A_v^{-2} \int_{-\infty}^{\infty} |g^{(v)}(x)|^2 \, dx < \infty \)

and

\( \|f-g\|_k < \varepsilon. \)

**Proof.** First, by Lemma 1, we can choose \( \eta > 0 \) so small that

\( \|f-f_\eta\|_k < \frac{1}{2} \varepsilon. \)

By Lemma 2, for arbitrary numbers \( 0 < L_1 < L \) there exists for an arbitrary \( k_0 > 0 \) an infinitely differentiable function \( \varphi(x) \) such that

\( \varphi(x) = \begin{cases} 
1, & -\frac{1}{2} \eta \leq x \leq L_1, \\
0, & \text{outside } (-\eta, L), 
\end{cases} \)

and
\begin{align*}
\sup_x |\psi^{(\nu)}(x)| & \leq C k_0^{\nu} A_\nu, \quad \nu = 0, 1, 2, \ldots.
\end{align*}

Define
\[
g(x) = \begin{cases} f_\eta(x) \psi(x), & x \geq -\eta, \\ 0, & x < -\eta. \end{cases}
\]

By (1.7) and (1.6)
\begin{equation}
A_{\nu-i} A_j \leq A_\nu, \quad j = 0, 1, \ldots, \nu.
\end{equation}

We also need the simple inequality
\begin{equation}
\binom{\nu}{j}^2 \leq \binom{2\nu}{2j}, \quad j = 0, 1, \ldots, \nu.
\end{equation}

For \(x \geq -\eta\) and \(\nu \geq 0\) we have, by (3.6) and (3.7),
\[
|g^{(\nu)}(x)| = \left| \sum_{j=0}^{\nu} \binom{\nu}{j} f^{(j)}(x + \eta) \psi^{(\nu-j)}(x) \right| \leq C A_\nu \sum_{j=0}^{\nu} \binom{\nu}{j} k_0^{\nu-j} A_j^{-1} |f^{(j)}(x + \eta)|.
\]

Hence, by Cauchy's inequality,
\[
|g^{(\nu)}(x)|^2 \leq C^2 A_\nu^2 \sum_{j=0}^{\nu} \binom{\nu}{j}^2 k_0^{2\nu-2j} \sum_{j=0}^{\nu} A_j^{-2} |f^{(j)}(x + \eta)|^2.
\]

But, by (3.8),
\[
\sum_{j=0}^{\nu} \binom{\nu}{j}^2 k_0^{2\nu-2j} \leq \sum_{i=0}^{2\nu} \binom{2\nu}{i} k_0^{2\nu-i} = (1 + k_0)^{2\nu}.
\]

Thus
\[
\int_{-\infty}^{\infty} |g^{(\nu)}(x)|^2 \, dx \leq C^2 A_\nu^2 (1 + k_0)^{2\nu} \|f\|_1^2
\]

and finally
\[
\sum_{\nu=0}^{\infty} \sum_{j=0}^{\nu} k^{-2\nu} A_\nu^{-2} \int_{-\infty}^{\infty} |f^{(j)}(x)|^2 \, dx \leq C^2 \|f\|_1^2 \sum_{\nu=0}^{\infty} (k^{-1}(1 + k_0))^{2\nu} < \infty
\]

if we choose \(k_0\) small enough.

By similar estimates
\[
\|f_\eta - g\|_k^2 \leq (C + 1)^2 \sum_{j=0}^{\infty} A_j^{-2} \int_{L_1} |f^{(j)}(x)|^2 \, dx \sum_{\nu=0}^{\infty} (k^{-1}(1 + k_0))^{2\nu}.
\]

Since \(\|f\|_1 < \infty\) this can be made arbitrarily small by taking \(L_1\) sufficiently large. Thus
\[
\|f_\eta - g\|_k < \frac{1}{2} \varepsilon
\]

and (3.5) is proved.

**Proof of Theorem 3.** By the Corollary of Theorem 2, every function \(e^{-\alpha x}, \Re \alpha > 0\), can be approximated arbitrarily closely in \(F_{A(0)}\), hence a
fortiori in $\mathcal{F}_{A(x^k), k > 1}$, by finite linear combinations of functions of $A$. Then, by Lemma 3, it is sufficient to show that if $g$ is a function of the type described in Lemma 3, there exists for every $\varepsilon > 0$ a function of the form

$$
Q(x) = \sum_{r=1}^{m} c_r e^{-\alpha_r x},
$$

where $\alpha_r$ are complex numbers with $\Re \alpha_r > 0$, such that

$$
\|g - Q\|_k < \varepsilon.
$$

For an arbitrary positive integer $n$, let

$$
P_n(x) = a_n^{-1}(1 - (1 - 2^{-x/L})^2)^n 2^{-x/L},
$$

where $a_n$ is chosen so that

$$
\int_{-L}^{\infty} P_n(x) \, dx = 1;
$$

we find

$$
a_n = \frac{L}{\log 2} \int_{-1}^{1} (1 - t^2)^n \, dt = \frac{2L}{\log 2} \frac{(2n)!!}{(2n + 1)!!}.
$$

The function

$$
Q(x) = \int_{-\infty}^{\infty} P_n(x - y) \, g(y) \, dy
$$

is of the form (3.9) and

$$
Q^{(\alpha)}(x) = \int_{-\infty}^{\infty} P_n(y) \, g^{(\alpha)}(x - y) \, dy.
$$

Hence for $x \geq 0$, by (3.10) and Schwarz's inequality,

$$
|g^{(\alpha)}(x) - Q^{(\alpha)}(x)|^2 \leq \int_{-L}^{\infty} P_n(y) \left| (g^{(\alpha)}(x) - Q^{(\alpha)}(x - y)) \right|^2 \, dy,
$$

and so

$$
\|g - Q\|_k^2 \leq \int_{-L}^{\infty} P_n(y) \, dy \sum_{r=0}^{\infty} k^{-2r} A_r^{-2} \int_{0}^{\infty} |g^{(\alpha)}(x) - g^{(\alpha)}(x - y)|^2 \, dx.
$$
By (3.4), we can make

\[ \sum_{r=N}^{\infty} k^{-2r} A_r^{-2} \int_{0}^{\infty} |g^{(r)}(x) - g^{(r)}(x-y)|^2 \, dx \]

arbitrarily small by taking \( N \) sufficiently large, independent of \( y \). Having fixed \( N \), since

\[ \lim_{y \to 0} \int_{-\infty}^{\infty} |g^{(r)}(x) - g^{(r)}(x-y)|^2 \, dx = 0 \]

also

\[ \sum_{r=0}^{N-1} k^{-2r} A_r^{-2} \int_{0}^{\infty} |g^{(r)}(x) - g^{(r)}(x-y)|^2 \, dx \]

is arbitrarily small if \( |y| \) is sufficiently small. Thus, there exists a \( \delta > 0 \) such that, independent of \( n \),

\[ I_1 = \int_{-\delta}^{\delta} P_n(y) \, dy \sum_{r=0}^{\infty} k^{-2r} A_r^{-2} \int_{0}^{\infty} |g^{(r)}(x) - g^{(r)}(x-y)|^2 \, dx < \frac{1}{4} \epsilon^2. \]

Finally, having fixed \( \delta \), we have to consider that part \( I_2 \) of the integral with respect to \( y \) in the right hand member of (3.11), where \( |y| \geq \delta \). By (3.4) it is sufficient to show that

\[ \lim_{n \to \infty} \left( \int_{-\delta}^{-L} P_n(y) \, dy + \int_{L}^{\delta} P_n(y) \, dy \right) = 0. \]

But that follows from

\[ \int_{-L}^{-\delta} P_n(y) \, dy = \frac{(2n+1)!!}{2 \cdot (2n)!!} \int_{-1}^{1-\delta/L} (1-t^2)^n \, dt \leq \text{const.} \cdot n! \cdot (1-(1-2\delta/L)^2)^n \]

and an analogous estimate of the second integral. Hence, for sufficiently large \( n \), \( I_2 < \frac{1}{4} \epsilon^2 \).

This completes the proof of Theorem 3.

4. The quasi-analytic case.

In the quasi-analytic case, (1.15) is not sufficient for a set \( A \) to be fundamental in \( \mathcal{L}_A^{(p)} \).

Starting with an increasing sequence \( \lambda \) of positive numbers such that (1.15) holds, suppose that a sequence \( A \) satisfying (1.6)–(1.8) and (1.12) is defined by
\begin{equation}
\prod_{j=1}^{\infty} (1 + \lambda_j^{-2} z^2) = \sum_{r=0}^{\infty} A_r^{-2} z^{2r}
\end{equation}

((1.7) is automatically fulfilled; see Boas [1, p. 24]). Then, as will appear below, there are reasons to expect the following to be true: The set

\[ \Lambda_a = \{ e^{-a \gamma j^2} : j = 1, 2, 3, \ldots \}, \quad a > 0, \]

is fundamental in \( \mathcal{L}_{A(a)} \), \( k > 1 \), if \( a < 2 \); but if \( a > 2k \), \( k \geq 1 \), there exists, for instance, an exponential function which does not belong to the closure in \( \mathcal{F}_{A(k)} \) of the span of \( \Lambda_a \). To motivate this conjecture we prove the following two theorems.

**Theorem 4.** Suppose that the sequence \( A \), defined by \((4.1)\), fulfills \((1.12)\). Then if \( a < 2 \) every function \( e^{-\alpha x} \), \( \text{Re} \alpha > 0 \), belongs to the closure in \( \mathcal{F}_{A(1)} \) of the span of \( \Lambda_a \).

**Proof.** By Theorem 2 it is sufficient to prove that \((2.3)\) holds. Applying Carlemann’s theorem to the entire function of exponential type

\[ \prod_{j=1}^{\infty} (1 - \lambda_j^{-2} z^2) = \sum_{r=0}^{\infty} (-1)^r A_r^{-2} z^{2r} \]

we get, since the first term in the right hand member of \((2.7)\) is bounded in this case (see Boas [1, p. 31]),

\[ a S(aR) = \sum_{\lambda_j \leq R} \lambda_j^{-1} \geq \sum_{\lambda_j \leq R} (\lambda_j^{-1} - \lambda_j R^{-2}) \]

\[ = \pi^{-1} \int_1^R y^{-2} \log \left( \sum_{r=0}^{\infty} A_r^{-2} y^{2r} \right) dy + O(1). \]

But for an arbitrary \( \beta, 0 < \beta < 1 \),

\[ (1 - \beta^2) \left( \sum_{r=0}^{\infty} A_r^{-1} (\beta y)^r \right)^2 \leq \sum_{r=0}^{\infty} A_r^{-2} y^{2r} \]

and hence

\[ \int_1^R y^{-2} \log \left( \sum_{r=0}^{\infty} A_r^{-2} y^{2r} \right) dy \geq 2\beta \int_1^R y^{-2} \log(\sum_{r=0}^{\infty} A_r^{-1} y^r) dy + O(1). \]

Thus

\[ a S(R) \geq 2\beta I(R) + O(1) \]

and so

\[ \lim_{R \to \infty} \frac{I(R)}{S(R)} \leq \frac{a}{2\beta} < 1 \]

if we choose \( \beta > \frac{1}{2} a \).
THEOREM 5. Let \( \lambda_j = \frac{1}{2} \pi j, j = 1, 2, 3, \ldots \). Then the corresponding set \( A^0_a \) is fundamental in \( L_{A(a)} \), \( k > 1 \), if \( a \leq 2 \), but if \( a > 2k \) there exists, for instance, an exponential function which does not belong to the closure in \( F_{A(a)} \), \( k \geq 1 \), of the span of \( A^0_a \).

PROOF. We have in this case

\[
A_v = 2^{-v} \left( (2v+1)! \right)^{\frac{1}{2}} = v! \left( 4\pi^{-1} v \right)^{\frac{1}{2}} (1 + \epsilon_v),
\]

where \( \epsilon_v \to 0 \) as \( v \to \infty \). If \( f \in L_{A(a)} \) it follows that

\[
\sup_{x \geq 0} |f^{(v)}(x)| \leq \text{const.} \gamma^v v!
\]

for an arbitrary \( \gamma > 1 \). Hence \( f(x) \) is the restriction to \([0, \infty)\) of a function \( f(z) \), holomorphic in the domain

\[
D: \begin{cases} |\text{Im} z| < 1, & \text{Re} z \geq 0, \\ |z| < 1, & \text{Re} z < 0, \end{cases}
\]

and bounded in every set

\[
\bar{D}_b: \begin{cases} |\text{Im} z| \leq b < 1, & \text{Re} z \geq 0, \\ |z| \leq b < 1, & \text{Re} z < 0. \end{cases}
\]

a. The case \( a \leq 2, k > 1 \).

We need the following lemma.

LEMMA 4. For \( f \in L_{A(a)} \),

\[
\lim_{\delta \to 0^+} \| f(x) - e^{-\delta x} f(x) \|_k = 0.
\]

PROOF. For

\[
g_\delta(x) = f(x) (1 - e^{-\delta x}), \quad x \geq 0, \quad \delta > 0,
\]

we find

\[
|g_\delta^{(v)}(x)| \leq \sum_{j=0}^{r-1} \binom{v}{j} \delta^{v-j} |f^{(j)}(x)| + (1 - e^{-\delta x}) |f^{(v)}(x)|.
\]

By Cauchy’s inequality,

\[
\frac{1}{2} |g_\delta^{(v)}(x)|^2 \leq \sum_{j=0}^{r-1} \binom{v}{j}^2 (2\delta)^{2v-2j} |f^{(j)}(x)|^2 + (1 - e^{-\delta x})^2 |f^{(v)}(x)|^2,
\]

and thus

\[
\frac{1}{2} \int_0^\infty |g_\delta^{(v)}(x)|^2 dx \leq \|f\|_1^2 \sum_{j=0}^{r-1} \binom{v}{j}^2 (2\delta)^{2v-2j} A_j^2 + \int_0^\infty (1 - e^{-\delta x})^2 |f^{(v)}(x)|^2 dx.
\]
Finally,
\[ \frac{1}{2} \| \Theta \|_{k}^{2} \leq \| f \|_{1}^{2} \sum_{r=0}^{\infty} k^{-2r} A_{r}^{-2} \sum_{j=0}^{r-1} \binom{r}{j}^{2} (2\delta)^{2r-2j} A_{j}^{2} + \]
\[ + \sum_{r=0}^{\infty} k^{-2r} A_{r}^{-2} \int_{0}^{\infty} (1 - e^{-\delta x})^{2} |f^{(r)}(x)|^{2} \, dx \]
\[ = S_{1}(\delta) + S_{2}(\delta). \]

For \( \delta < \frac{1}{4} \), by (3.8) and (4.3),
\[ \| f \|_{1}^{-2} S_{1}(\delta) \leq (4\delta)^{2} \sum_{r=0}^{\infty} k^{-2r} \sum_{j=0}^{\infty} 1/(2j)! , \]
and so
\[ \lim_{\delta \to 0+} S_{1}(\delta) = 0 . \]

By dominated convergence,
\[ \lim_{\delta \to 0+} S_{2}(\delta) = 0 , \]
and thus the proof of Lemma 4 is complete.

To prove Theorem 5 in the case \( a \leq 2 \) we perform the conformal mapping \( w = e^{-t \alpha z} \). Then \( D_{b} \) is mapped onto a Jordan region \( \Omega_{b} \) in the \( w \)-plane with 0 on the boundary. Let for \( 0 < \delta < 1 \)
\[ F(w) = \begin{cases} \alpha^{2} f(z), & w \neq 0 , \\ 0 , & w = 0 . \end{cases} \]

Since \( F(w) \) is holomorphic in \( \Omega_{b} \) and continuous in \( \partial \Omega_{b} \), \( F(w) \) can be approximated uniformly and arbitrarily closely in \( \partial \Omega_{b} \) by a polynomial in \( w \) without constant term. But every function \( w^{m-1+i\delta} \), where \( m \) is a positive integer, can be approximated in the same way and this implies that, for an arbitrary \( \varepsilon > 0 \), there exists a function
\[ P(w) = \sum_{j=1}^{N} a_{j} w^{j-\delta} \]
such that
\[ \max_{\partial \Omega_{b}} |F(w) - P(w)| < \varepsilon . \]

This yields, for
\[ h(z) = e^{-\alpha \delta z} f(z) \quad \text{and} \quad Q(z) = \sum_{j=1}^{N} a_{j} e^{-j \alpha z} , \]
the inequality
\[ \max_{\partial \Omega_{b}} |h(z) - Q(z)| < C \varepsilon \]
and, for some constant \( M \) independent of \( \varepsilon \),
\[ |Q(z)| < Me^{-\text{log}(z)} , \quad z \in \overline{D}_b . \]

By Lemma 4 it is now sufficient to prove that \(\|h - Q\|_k\) can be made arbitrarily small.

By Cauchy's estimates,
\[
|Q^{(r)}(x)| \leq M \nu! b^{r-\nu} e^{-\text{log}(x-1)} , \quad x \geq 0 ,
\]
and hence, choosing \(b\) so that \(kb > 1,\)
\[
\sum_{r=0}^{\infty} k^{-2r} A^{-2} \int_{R}^{\infty} |Q^{(r)}(x)|^2 \, dx \leq \frac{M^2}{\pi \alpha} e^{-\text{log}(R-1)} \sum_{r=0}^{\infty} (kb)^{-2r} .
\]
This, and also
\[
\sum_{r=0}^{\infty} k^{-2r} A^{-2} \int_{R}^{\infty} |h^{(r)}(x)|^2 \, dx ,
\]
is arbitrarily small if \(R\) is sufficiently large. Finally, for \(x \geq 0,\)
\[
|h^{(r)}(x) - Q^{(r)}(x)| \leq C b^{-\nu!} \varepsilon ,
\]
and hence
\[
\sum_{r=0}^{\infty} k^{-2r} A^{-2} \int_{0}^{R} |h^{(r)}(x) - Q^{(r)}(x)|^2 \, dx < RC^2 \varepsilon^2 \sum_{r=0}^{\infty} (kb)^{-2r} ,
\]
which is arbitrarily small if \(\varepsilon\) is sufficiently small. This proves Theorem 5 in the case \(a \leq 2.\)

b. The case \(a > 2k, \quad k \geq 1.\)

Suppose that for a certain function \(f\) in \(L_{A^{(k)}}\) there exists a sequence \(\{Q_n(z)\}_{n=1}^{\infty}\), where \(Q_n(x)\) belongs to the span of \(A_{a,0}^n\), such that
\[
(4.5) \quad \lim_{n \to \infty} \|f - Q_n\|_k = 0 .
\]
Hence
\[
\|Q_n\|_k \leq \text{const}.
\]
uniformly in \(n\) and so for an arbitrary \(\gamma > 1\)
\[
\sup_{x \geq 0} |Q_n^{(\nu)}(x)| \leq \text{const.} (k\gamma)'' \nu! .
\]
Thus, on every compact subset of the half strip
\[
(4.6) \quad |\text{Im} z| < 1/k, \quad \text{Re} z > 0 ,
\]
we have
\[
\sup |Q_n(z)| \leq \text{const}.
\]
uniformly in n. The family \( \{Q_n(z)\}_{n=1}^{\infty} \) is thus normal in the half strip (4.6). Since, by (4.5),
\[
\lim_{n \to \infty} Q_n(x) = f(x)
\]
uniformly on \([0, \infty)\), we infer that
\[
\lim_{n \to \infty} Q_n(z) = f(z)
\]
uniformly on every compact subset of the half strip (4.6). But \(Q_n(z)\) is periodic with period \(4i/a\) and then this is true also for \(f(z)\), if the width of the half strip is greater than \(4/a\), that is, if \(a > 2k\). However, there are for instance exponential functions for which this is not true. This completes the proof of Theorem 5.

REFERENCES