CLASSIFICATION AND DEFORMATION
OF RIEMANNIAN SPACES

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A promising new field of research lies in the classification theory of
Riemannian spaces with respect to the existence or nonexistence of
harmonic functions with various boundedness properties. For 2-spaces
harmonicity is known to be invariant not only under isometries but also
under conformal mappings. In contrast, there is no conformal invariance
for higher dimensions. This is exemplified by the following rather sur-
prising phenomena:

(a) A ball can be endowed with a Riemannian metric, conformally
equivalent to the Euclidean one, which makes the ball parabolic.
(b) A compact locally Euclidean space punctured at a point can be
endowed with a conformally equivalent Riemannian metric which
makes the resulting space hyperbolic.
(c) The classes \( O_G \), \( O_{HD} \), and \( O_{KD} \) are not quasi-conformally, and not
even conformally invariant. They are, however, quasi-isometrically
invariant.

We start by recalling in Section 1 the definitions of the null classes \( O_G \),
\( O_{HD} \), and \( O_{KD} \). Making use of function spaces introduced in Section 2
we characterize in Sections 3–5 these null classes by equivalence rela-
tions. In Section 6 we introduce the Dirichlet mappings and show by
means of the equivalence relations that the null classes are invariant
under Dirichlet mappings. In particular, quasi-conformal mappings of
Riemannian 2-spaces are shown to be Dirichlet mappings. That the
invariance under quasi-conformal mappings is lost for higher dimensions
is established in Section 7, which also contains the proofs of the above
statements (a) and (b). The paper closes in Section 8 with the result
that quasi-isometries are Dirichlet mappings, hence the quasi-isometric
invariance of \( O_G \), \( O_{HD} \), and \( O_{KD} \).

The work was sponsored by the U.S. Army Research Office—Durham, Grant DA-
AROD-31-124-G742, University of California, Los Angeles.
Received July 25, 1966.
1. Null classes of Riemannian spaces.

1. Let $V$ be a noncompact Riemannian space, i.e., a connected, orientable, and $C^\infty$ $n$-manifold with a positive definite metric tensor $g_{ij}$. The inverse matrix of the $n \times n$ matrix $(g_{ij})$ is denoted by $(g^{ij})$ and the determinant of $(g_{ij})$ by $g$. (For fundamentals of Riemannian spaces see, e.g., de Rham [19], Hodge [10].)

We consider the class $H(V)$ of harmonic functions in $V$, i.e., solutions of the self-adjoint second order elliptic partial differential equation

$$
\Delta u = -\frac{1}{g^i} \frac{\partial}{\partial x^i} \left( g^i g^{ij} \frac{\partial u}{\partial x^j} \right) = 0
$$

in $V$. (For basic properties of solutions see, e.g., Miranda [14], Feller [6], Itô [12], Hörmander [11].)

For a set $E \subset V$ we denote by $H(E)$ the class of functions $u$ harmonic in some open sets $\mathcal{O}_u$ containing $E$. For an open set $\mathcal{O} \subset V$ the symbol $H^c(\mathcal{O})$ will be used for the class of functions harmonic in $\mathcal{O}$ and with continuous extensions to $\overline{\mathcal{O}}$:

$$
H^c(\mathcal{O}) = H(\mathcal{O}) \cap C(\overline{\mathcal{O}})
$$

2. Let $\{V_m\}_{m=0}^\infty$ with $\overline{V}_m \subset V_{m+1}$ be an exhaustion of $V$ by regular subregions $V_m$ (Sario–Schiffer–Glasner [24]). For $m > 0$ we consider the functions $u_m \in C(V)$ defined by

$$
u_m|_{\overline{V}_0} = 0, \quad u_m|_{V - V_m} = 1, \quad u_m|_{\overline{V}_m - V_0} \in H^c(V_m - \overline{V}_0).
$$

The sequence $\{u_m\}_{m=0}^\infty$ is monotone decreasing and converges uniformly in compact subsets to a nonnegative function $u \in C(V)$ with

$$
u|_{\overline{V}_0} = 0, \quad v|_{V - V_0} \in H^c(V - \overline{V}_0).
$$

There are only two cases: either $u \equiv 0$ on $V$ or $u|_{V - \overline{V}_0} > 0$. In the former case we call $V$ parabolic, and denote the class of parabolic Riemannian spaces by $O_G$. The notation is in reference to the fact that these spaces can also be characterized by the nonexistence of Green’s functions of (1) on $V$ (cf. Itô [12], Glasner [9]). The spaces with $u \equiv 0$ will be called hyperbolic.

3. The Dirichlet integral of a function $f$ on $V$ is, by definition,

$$
D(f) = \int_\overline{V} |\text{grad} f|^2 dV = \int_\overline{V} df \wedge *df,
$$

where
\begin{equation*}
|\text{grad} f|^2 = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}
\end{equation*}

is supposed to exist a.e. and \( dV = g^1 dx^1 \wedge \ldots \wedge dx^n \) (cf. Duff [5]).

We denote by \( HD(V) \) the class of functions \( u \in H(V) \) with finite Dirichlet integrals \( D(u) \), and by \( O_{HD} \) the class of Riemannian spaces \( V \) that do not carry nonconstant functions in \( HD(V) \).

We are also interested in the class \( KD(V) \) of functions \( u \) in \( HD(V) \) with vanishing flux

\begin{equation*}
\int_{\gamma} \ast du = 0
\end{equation*}

across all dividing \((n-1)\)-cycles \( \gamma \) in \( V \). The corresponding null class is denoted by \( O_{KD} \).

4. It is known that

\( O_G \subset O_{HD} \subset O_{KD} \)

(Sario–Schiffer–Glasner [24]), the inclusions being strict. To investigate the invariance of these classes under various mappings of \( V \) we shall characterize them by equivalences of certain function spaces which we shall now introduce.

For further classification problems of Riemannian spaces reference is here made to the doctoral dissertations of Smith [25], Glasner [9], Breazeal [3], and Ow [17].

2. Norms and function spaces.

5. Given an exhaustion \( \{V_m\}_{m=0}^{\infty} \) of \( V \), we shall make use of the quasinorm

\begin{equation*}
[f]_\infty = \sum_{m=0}^{\infty} \frac{1}{2^m} \sup_{V_m} \frac{|f|}{1 + |f|}.
\end{equation*}

Convergence in this norm is clearly equivalent to compact convergence, i.e., convergence in every compact subset of \( V \). Uniform convergence in all of \( V \) is given by the norm

\begin{equation*}
\|f\|_\infty = \sup_{V} |f|,
\end{equation*}

whereas the Dirichlet convergence is provided by the seminorm

\begin{equation*}
\|f\| = (D(f))^1.
\end{equation*}

Two combinations will be useful:

\begin{equation*}
\langle f \rangle = [f]_\infty + \|f\|
\end{equation*}
corresponding to both compact and Dirichlet convergence, and
\[ \langle \langle f \rangle \rangle = \| f \|_\infty + \| f \| \]
furnishing uniform and Dirichlet convergence.

6. Given a parametric ball \( B \) with the coordinate system \( x \), we denote by \( ACL(B) \) the class of functions in \( B \) that are absolutely continuous on almost all lines parallel to the coordinate axes. Analogously, \( ACL(V) \) shall designate the class of functions \( f \) on \( V \) with \( f|B \in ACL(B) \) for every parametric ball \( B \) and every coordinate system in \( B \).

Of basic importance in our study will be the function space
\[ D = \{ f \mid f \in C(V) \cap ACL(V), \| f \| < \infty \}. \]
It is easily seen that \( \partial f/\partial x^i, i = 1, \ldots, n \), exists a.e. and \( df = (\partial f/\partial x^i)dx^i \) a.e.

The space \( D \) is closed in the norm \( \langle \cdot \rangle \), hence also in \( \langle \langle \cdot \rangle \rangle \) (see Nakai [15]). It is also closed under the operations \( f \cup g = \max(f, g) \) and \( f \cap g = \min(f, g) \), with \( f, g \in D \) (Nakai [16]). The subspace \( D_\infty = C_\infty \cap D \) is dense in terms of \( \langle \cdot \rangle \) (cf. [15]).

We are also interested in the subspaces
\[ D_0 = \{ f \mid f \in D, \text{supp} f \text{ compact} \}, \]
\[ D_1 = \{ f \mid f \in D, \text{supp} df \text{ compact} \}. \]
Again the spaces \( D_{i\infty} = C_\infty \cap D_i, i = 0, 1 \), are dense in \( D_i \). As a consequence Stokes’ formula can be used freely for functions in \( D \) and \( D_i \).

3. Characterization of \( O_G \).

7. Denote by \( \overline{D}_0 \) the closure of \( D_0 \) in \( D \) with respect to \( \langle \cdot \rangle \). We shall prove (cf. Royden [20], [22], Nakai [15], Glasner [9]):

**Theorem 1.** \( V \in O_G \) if and only if \( D = \overline{D}_0 \).

For functions \( u_m \) in No. 2, Stokes’ formula gives
\begin{equation}
D(u_m) = D(u_{m+p}) + D(u_{m+p} - u_m)
\end{equation}
for \( p \geq 0 \), and we have \( D(u - u_m) \to 0 \). Since \( [u - u_m]_\infty \to 0 \), it follows that \( \langle u - u_m \rangle \to 0 \).

Suppose \( V \in O_G \). We know from No. 2 that \( \langle u_m \rangle \to 0 \), and we set \( v_m = 1 - u_m \). Given \( f \in D \), we are to show the existence of a sequence \( \{ f_m \} \subset D_0 \) such that \( \langle f - f_m \rangle \to 0 \).

First we consider the case where \( f \) is bounded, \( |f| < M < \infty \). Obviously \( [f - v_m f]_\infty \to 0 \) with \( \{ v_m f \} \subset D_0 \) and for the Dirichlet norm we have
\[ D(f - v_m f) = D(u_m f) = \int \left| \text{grad} u_m f \right|^2 dV \]
\[ \leq 2 \int \left( \sup_V u_m \right)^2 |\text{grad} f|^2 dV + 2 \int | \text{grad} u_m |^2 \left( \sup_V |f| \right)^2 dV. \]

Here the second term is dominated by \(2M^2D(u_m) \to 0\). We decompose the integral \(I_V\) in the first term into \(I_K\) and \(I_{V - K}\), with \(K\) a compact set, and have

\[ \lim \sup_m I_V \leq \lim \sup_m I_K + \lim \sup_m I_{V - K} \leq D_{V - K}(f). \]

For a sufficiently large \(K\) this is arbitrarily small, and we obtain \(|f - v_m f| \to 0\), hence \(\langle f - v_m f \rangle \to 0\).

If \(f \in \mathcal{D}\) is unbounded, we approximate it by the truncated bounded functions

\[ f_m = \max(\min(f, m), -m) \in \mathcal{D}. \]

Obviously \([f - f_m] \to 0\) and, by virtue of the continuity of the norm \(|\cdot|\) in the region, \(|f - f_m| \to 0\). A fortiori \(\langle f - f_m \rangle \to 0\), and since each \(f_m\) is a limit in \(\langle \cdot \rangle\) of functions in \(\mathcal{D}_0\), the same is true of \(f\). The proof of the necessity of \(\mathcal{D} = \overline{\mathcal{D}}_0\) is herewith complete.

8. To establish the sufficiency, note that \(\mathcal{D} = \overline{\mathcal{D}}_0\) implies \(1 \in \mathcal{D}_0\), that is, the existence of a sequence of function \(\varphi_m \in \mathcal{D}_0\) with \(\langle 1 - \varphi_m \rangle \to 0\).

It follows that

\[ D(u) = \lim_m D(u_m) = - \int_{\partial \nabla_0} \ast \overline{d} u = - \lim_m \int_{\partial \nabla_0} \varphi_m \ast \overline{d} u \]
\[ = \lim_m D(\varphi_m, u) \leq \lim_m (D(\varphi_m))^\top(D(u))^\top = 0, \]

and we have \(u \equiv 0, \ V \in O_G\).

4. Characterization of \(O_{HD}\).

9. In view of the inclusion \(O_G \subset O_{HD}\) we may assume that \(V\) is hyperbolic. We shall need the following auxiliary result (cf. Royden [20], Brelot [4], Nakai [16]):

**Lemma 1.** Assume that \(V \notin O_G\). The vector space \(\mathcal{D}\) is the direct sum of vector subspaces \(HD\) and \(\overline{\mathcal{D}}_0\):

\[ \mathcal{D} = HD + \overline{\mathcal{D}}_0. \]

Spaces \(HD\) and \(\overline{\mathcal{D}}_0\) are also orthogonal in terms of the inner product \(D(\cdot, \cdot)\).
Given \( f \in \mathcal{D} \) we know from No. 6 that \( f^+ = \max(f, 0) \) and \( f^- = \max(-f, 0) \) are in \( \mathcal{D} \). For this reason we may assume that \( f \geq 0 \). Then the function \( h_m \in \mathcal{C}(V) \) defined by

\[
\left. h_m \right| \bar{V}_m \in \mathcal{H}^e(V_m), \quad \left. h_m \right| V - V_m = f
\]
is nonnegative. On setting \( g_m = f - h_m \) we obtain

\[
D(f) = D(h_m) + D(g_m)
\]
and

\[
D(h_m) = D(h_{m+p}) + D(h_m - h_{m+p}).
\]

We infer that \( \{h_m\} \), and consequently \( \{g_m\} \), is a Cauchy sequence in \( || \cdot || \).

10. We shall show that the same is true in \( [\cdot]_\infty \). In fact,

\[
\int_{\bar{V}_m} h_m * du_m = \int_{\bar{V}_m} f * du_m + \int_{\bar{V}_m - \partial \bar{V}_0} g_m * du_m,
\]
and for \( p \in V_0 \) we have by Harnack’s inequality and by (4)

\[
ch_m(p)D(u_m) \leq \min_{\partial \bar{V}_0} fD(u_m) + (D(f))^i(D(u_m))^i,
\]

where \( c \) is a constant \( \neq 0 \). In view of \( 0 < D(u) \leq D(u_m) \) this implies

\[
ch_m(p) \leq \min_{\partial \bar{V}_0} f + (D(f)/D(u))^i,
\]
and therefore \( |h_m(p)| < N \) for some finite \( N \). It follows again by Harnack’s inequality that \( \{h_m\} \) is uniformly bounded in \( V \), and therefore a normal family. There exists a subsequence, again denoted by \( \{h_m\} \), and an \( h \in H(V) \) such that \( [h - h_m]_\infty \to 0 \).

11. In summary we have \( \langle h - h_m \rangle \to 0 \) and \( h \in HD \). The function \( g = f - h \) has the corresponding property \( \langle g - g_m \rangle = \langle h - h_m \rangle \to 0 \), and therefore \( g \in \mathcal{D}_0 \). Moreover, (4) yields for \( m \to \infty \)

\[
D(f) = D(h) + D(g).
\]

We thus have \( h \perp g \), and it follows easily that \( HD \perp \mathcal{D}_0 \).

To prove the uniqueness of the orthogonal decomposition (3) so established suppose \( f \equiv 0 \). By the above relation \( h = c \), a constant, and \( g = -c \). If \( c \neq 0 \), then \( c \in \mathcal{D}_0 \) implies \( 1 \in \mathcal{D}_0 \), and we have proved in No. 8 that this entails \( V \in O_G \), contrary to our assumption. We conclude that \( c = 0 \), and the uniqueness is established.

12. Let \( R \) be the real number field. We are ready to prove (cf. Royden [20], [22], Nakai [15], [16]):
THEOREM 2. \( V \in O_{HD} \) if and only if \( \mathcal{D} = \overline{\mathcal{D}}_0 \) (mod \( R \)).

If \( V \in O_G \), there is nothing to prove since \( \mathcal{D} = \overline{\mathcal{D}}_0 \). We therefore assume that \( V \notin O_G \).

Suppose \( V \in O_{HD} \). Then \( \mathcal{D} = R + \overline{\mathcal{D}}_0 \), as maintained. Conversely, if this equality holds, then \( HD = R \).

5. Characterization of \( O_{KD} \).

13. Let \( \overline{\mathcal{D}}_1 \) be the closure in \( \mathcal{D} \) of \( \mathcal{D}_1 \) with respect to \( || \cdot || \). Lemma 1 has the following counterpart for \( KD \) (cf. Ahlfors [1], Royden [22]):

**Lemma 2.** The space \( \mathcal{D} \) has the orthogonal decomposition

\[
\mathcal{D} = KD + \overline{\mathcal{D}}_1, \quad KD \cap \overline{\mathcal{D}}_1 = R
\]

in terms of \( D(\cdot, \cdot) \).

We shall first show that \( \overline{\mathcal{D}}_1 \perp KD \). Let \( \varphi \perp \overline{\mathcal{D}}_1 \). Then by virtue of \( \mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \overline{\mathcal{D}}_1 \) we have \( \varphi \perp \mathcal{D}_0 \) and, by continuity, \( \varphi \perp \overline{\mathcal{D}}_0 \). By (3) this implies \( \varphi \in HD \).

Let \( \gamma \) be a dividing \((n-1)\)-cycle in \( V \). There exists an open regular subset \( \Omega \) of \( V \) and a function \( \psi \in C^\infty(V) \) with the following properties:

(\( \alpha \)) \( \partial \Omega = \gamma \cup \gamma' \),

(\( \beta \)) \( \gamma \) is homologous to \( \gamma' \),

(\( \gamma \)) \( \psi \) is constant in each component of \( V - \Omega \),

(\( \delta \)) \( \psi|_\gamma = 1, \psi|_{\gamma'} = 0 \).

Clearly \( \psi \in \mathcal{D}_1 \), and

\[
0 = D(\varphi, \psi) = \int_\gamma *d\varphi.
\]

Consequently \( \varphi \in KD \).

14. To prove the orthogonality \( KD \perp \overline{\mathcal{D}}_1 \) we observe that \( D(\varphi, \psi) = 0 \) for \( \varphi \in KD \), \( \psi \in \mathcal{D}_1 \), and the statement follows by continuity.

15. It remains to prove that \( KD + \overline{\mathcal{D}}_1 \) actually is \( \mathcal{D} \). To this end let \( \mathcal{D}^- \) and \( \mathcal{D}_1^- \) be the completions of \( \mathcal{D} \) and \( \mathcal{D}_1 \) in \( || \cdot || \). Then \( \mathcal{D}^- \cap C = \mathcal{D} \) and \( \mathcal{D}_1^- \cap C = \mathcal{D}_1 \). For the Hilbert space \( \mathcal{D}^- \) considered mod \( R \) we have the decomposition

\[
\mathcal{D}^- = \mathcal{D}_1^- + \mathcal{D}_1^-.
\]

We repeat the argument on \( KD \) in Nos. 13, 14 by replacing \( \mathcal{D} \) by \( \mathcal{D}^- \) and \( \mathcal{D}_1 \) by \( \mathcal{D}_1^- \) to obtain \( \mathcal{D}_1^- = KD \), and consequently

\[
\mathcal{D}^- = KD + \mathcal{D}_1^-.
\]
On taking intersections with $C$ we arrive at
\[ \mathcal{D} = KD + \overline{\mathcal{D}}_1. \]

The proof of Lemma 2 is herewith complete.

16. We are ready to characterize the class $O_{KD}$.

**THEOREM 3.** $V \in O_{KD}$ if and only if $\mathcal{D} = \overline{\mathcal{D}}_1$.

Since the orthogonality $KD \perp \overline{\mathcal{D}}_1$ is in $D(\cdot, \cdot)$, the null space is $R$. If $KD = R$, then by (6) $\mathcal{D} = \overline{\mathcal{D}}_1$, and conversely.

6. Dirichlet mappings.

17. Let $V, \overline{V}$ be Riemannian $n$-spaces. To study invariances of our null classes we introduce:

**DEFINITION.** A homeomorphism $T: V \to \overline{V}$ is a Dirichlet mapping provided that
\[ f \circ T \in \mathcal{D}(V) \text{ if and only if } f \in \mathcal{D}(\overline{V}) \]
and there exists a constant $K \geq 1$ with the property
\[ K^{-1}D(f) \leq D(f \circ T) \leq KD(f) \]
for all $f \in \mathcal{D}(\overline{V})$.

From Theorems 1–3 we conclude immediately:

**THEOREM 4.** The classes $O_G$, $O_{HD}$, and $O_{KD}$ are invariant under Dirichlet mappings.

In fact, the induced mapping $T^*: \mathcal{D}(\overline{V}) \to \mathcal{D}(V)$ given by $T^*f = f \circ T$ is a homeomorphism in $\| \cdot \|$ and also in $[\cdot ]_\infty$. For this reason
\[ T^*(\mathcal{D}_0(\overline{V})) = \mathcal{D}_0(V), \quad T^*(\mathcal{D}_1(\overline{V})) = \mathcal{D}_1(V). \]

18. After these preparations we are ready to turn to quasi-conformal mappings. For $p, q \in V$ we set
\[ d(p, q) = \inf_\alpha \int (g_{ij}dx^i dx^j)^{1/4}, \]
where the infimum is taken among all rectifiable arcs $\alpha$ from $p$ to $q$. In terms of $d$ and similarly defined $\overline{d}$ in $\overline{V}$ we write
\[ l(p, r) = \inf_{d(p, q) = r} \overline{d}(T(p), T(q)). \]
\[ L(p, r) = \sup_{d(p, q) = r} \bar{d}(T(p), T(q)). \]

**Definition.** A homeomorphism \( T: V \to \bar{V} \) is quasi-conformal if there exists a constant \( K \geq 1 \) such that
\[ \lim_{r \to 0} \frac{L(p, r)}{l(p, r)} \leq K \]
for all \( p \in V \).

By a simple generalization of the results of Gehring [7], [8] and Väisälä [26] we see that a quasi-conformal mapping is ACL in terms of local parameters of \( V \) and \( \bar{V} \) and that it is totally differentiable a.e. Moreover at any \( p \in V \),
\[ I(p)^n \leq K^{n-1} J(p), \]
where
\[ I(p) = \lim_{r \to 0} \sup L(p, r)/r \]
and
\[ J(p) = \lim_{r \to 0} \bar{V}(TB_r)/V(B_r). \]
Here \( B_r \) is the geodesic \( r \)-ball at \( p \), and \( \bar{V}(TB_r), V(B_r) \) are the volumes of \( TB_r \) and \( B_r \) respectively. At a point \( p \) where \( T \) is totally differentiable, \( J \) is the Jacobian and we have \( J > 0 \) a.e. Moreover if \( E \subset V \) is measurable, then so is \( TE \), and its volume is
\[ \bar{V}(TE) = \int_E J \, dV. \]

Corresponding properties, with the same \( K \), are possessed by the inverse mapping \( T^{-1} \).

19. In the remainder of Section 6 we restrict our attention to spaces of dimension 2. We shall show (Pfluger [18], Royden [21, 22], Nakai [15]):

**Theorem 5.** The classes \( O_G, O_{HD}, \) and \( O_{KD} \) of Riemannian 2-spaces are invariant under quasi-conformal mappings.

For the proof we shall first consider functions \( f \in \mathcal{D}^\infty(\bar{V}) \) and show that they satisfy conditions (7) and (8) (cf. Künzi [13]). Clearly \( f \circ T \) is in \( \mathcal{D}(V) \) and totally differentiable a.e. We consider only points \( p \in V \) with this property. We shall show that
\[ |\text{grad} F|^2 \leq |\text{grad} f|^2 I^2(p), \]
where \( F = f \circ T \). If \( \text{grad} F = 0 \), then this is trivially true. We shall assume that \( |\text{grad} F| \neq 0 \).
At \( p \) and \( \vec{p} = T(p) \) choose coordinate systems \( x \) and \( \tilde{x} \) with \( g_{ij} = \delta_{ij} \) and \( \tilde{g}_{ij} = \delta_{ij} \). For \( F_i = \partial F/\partial x^i \) at \( p \) we have \( F_i dx^i = f_i d\tilde{x}^i \). Since \( \text{grad} F \equiv 0 \) we can choose the \( dx^i \) such that \( F_i/\sum dx^i = F_2/\sum dx^2 \). Then by the Schwarz inequality

\[
(\sum F_i^2) (\sum dx^2) = (F_i dx^i)^2 \leq (\sum f_i^2)(\sum d\tilde{x}^i)^2.
\]

Since \( g_{ij} = \delta_{ij} \), we have \( \sum f_i^2 = |\text{grad} f|^2 \) and \( \sum F_i^2 = |\text{grad} F|^2 \). Similarly \( \sum (dx^i)^2 = ds^2 \) and \( \sum (d\tilde{x}^i)^2 = d\tilde{s}^2 \). But \( d\tilde{s}^2/\sum ds^2 \leq I^2(p) \), and (10) follows.

20. By virtue of \( I^2 \leq KJ \) we obtain

\[ |\text{grad} f \circ T|^2 \leq K |\text{grad} f|^2 J(p). \]

An integration with respect to \( dV \) yields

\[ D(f \circ T) \leq KD(f). \]

By a similar reasoning we conclude that

\[ K^{-1} D(f) \leq D(f \circ T). \]

Properties (7) and (8) have thus been established for \( f \in D^\infty(\tilde{V}) \).

21. Suppose now \( f \in D(\tilde{V}) \). Then there exists a sequence of functions \( f_m \in D^\infty(\tilde{V}) \) such that \( \langle f - f_m \rangle \rightarrow 0 \) if \( m \rightarrow \infty \). Since \( f_m \circ T \in D(V) \),

\[
K^{-1} D(f_m) \leq D(f_m \circ T) \leq KD(f_m),
\]

and

\[
K^{-1} D(f_m - f_{m+p}) \leq D(f_m \circ T - f_{m+p} \circ T) \leq KD(f_m - f_{m+p}),
\]

we see that

\[ \langle f \circ T - f_m \circ T \rangle \rightarrow 0. \]

Thus (7) holds: \( f \circ T \in D(V) \). On letting \( m \rightarrow \infty \) we obtain (8).

7. Conformal noninvariance.

22. Broadly speaking, the reason for the invariance under a quasi-conformal mapping \( T \) in the case \( n = 2 \) is that \( |\text{grad} f \circ T|^2 \) is, in essence, divided by \( I^2 \), while \( dV \) is multiplied by \( J \), and by virtue of the relation \( I^n \leq K^{n-1} J \) these changes compensate in \( D(f \circ T) \). For \( n > 2 \) this compensation no longer takes place and the above proof breaks down. We shall now show that the difficulty is in the very nature of the phenomenon: there does not exist even conformal invariance.
We start by constructing two special Riemannian spaces. The first one provides us with a proof of statement (a) of the introduction.

In Euclidean $n$-space with $n > 2$, set $|x|^2 = \sum_1^n (x^i)^2$.

**Example 1.** The ball $B: |x| < 1$ becomes parabolic when endowed with the metric $g_{ij}(x) = \lambda(x) \delta_{ij}$, where $\lambda \in C^\infty$, $\lambda > 0$ in $B$ and

\[
\lambda(x) = |x|^{(2-2n)/(n-2)} (1 - |x|)^4/(n-2)
\]

in $\frac{1}{2} \leq |x| < 1$.

For the proof note first that equation (1) takes the form

\[
\Delta u = -\lambda(x) \sum_{i=1}^n \frac{\partial}{\partial x_i} \lambda(x) \frac{\partial u}{\partial x_i}.
\]

We exhaust $V = B$ by

\[
V_m : |z| < 1 - \frac{1}{2 + m}
\]

and set

\[
u_m(x) = \frac{1}{m(1 - |x|)} - \frac{2}{m}
\]

in $V_m - V_0$. Then

\[
\lambda(x) \frac{\partial u_m}{\partial x_i} = \frac{x_i}{m |x|^n}
\]

and

\[-\lambda(x) \Delta u_m = \sum_{i=1}^n -\frac{nx_i^2 + |x|^2}{m |x|^{n+2}} = 0.
\]

Thus $u_m$ is harmonic, the limiting function is $u \equiv 0$, and therefore $V \in O_G$.

23. We turn to the proof of statement (b) of the introduction.

**Example 2.** Let $B$ be a parametric ball $|x| < 1$ of a compact locally Euclidean space $V_0$ with $n > 2$. The space $V = V_0 - p$, with $p$ the center of $B$, becomes hyperbolic when endowed with a Riemannian $C^\infty$ metric in $V$ such that $g_{ij}(x) = \lambda(x) \delta_{ij}$ in $V$ with $\lambda > 0$ in $V$ and

\[
\lambda(x) = |x|^{(2-2n)/(n-2)}
\]

in $B - p$.

In $B - p$, equation (13) is again valid. We choose

\[
B_m : |x| < \frac{1}{1 + m}, \quad V_m = V - \overline{B_m},
\]

and
in $\bar{V}_m - V_0$. Then
\[
\lambda^{n-2} \frac{\partial u_m}{\partial x^i} = -\frac{c x^i}{|x|^m}
\]
with $c$ a constant, and
\[
-\lambda^n \Delta u_m = c \sum_{i=1}^{n} \frac{-n x_i^2 + |x|^2}{|x|^{n+2}} = 0
\]
The limiting function is $u = 1 - |x| \neq 0$, and $V \notin O_G$.

24. We are ready to prove that Theorem 5 is no longer true for higher dimensions:

**Theorem 6.** The classes $O_G$, $O_{HD}$, and $O_{KD}$ are not conformally invariant.

Let $V$ be the ball $B$: $|x| < 1$ with the Euclidean metric $g_{ij} = \delta_{ij}$. For $\bar{V}$ take the same ball with the metric $g_{ij} = \lambda \delta_{ij}$ of Example 1. We shall show that the mapping $T: V \to \bar{V}$ induced by the identity mapping of the base space $B$ is conformal.

Let $p, q \in V$ and, in the notation of No. 18, $d(p,q) = r$. Then
\[
\bar{d}(T(p), T(q)) = \lambda(p)r + \varepsilon(p,q),
\]
where $\varepsilon(p,q) \to 0$ as $d(p,q) \to 0$. We have
\[
L(p, q) = \lambda(p)r + \sup_{d(p,q) = r} \varepsilon(p,q),
\]
\[
l(p, q) = \lambda(p)r + \inf_{d(p,q) = r} \varepsilon(p,q),
\]
and consequently
\[
\lim_{r \to 0} \frac{L(p,r)}{l(p,r)} = 1,
\]
as claimed.

Clearly $V \notin O_G$, $O_{HD}$, $O_{KD}$, whereas $\bar{V} \in O_G \subset O_{HD} \subset O_{KD}$. Theorem 6 is herewith established.

8. Quasi-isometric invariance.

25. Although our null classes for $n > 2$ lack quasi-conformal invariance, they have another important invariance property, associated in a perhaps more natural manner with the Riemannian metric.

**Definition.** A homeomorphism $T: V \to \bar{V}$ is a quasi-isometry if there exists a constant $K > 1$ such that
(17) \[ K^{-1}r \leq l(p,r) \leq L(p,r) \leq Kr \]

for all \( p \in V \).

A quasi-isometry is clearly a quasi-conformal mapping and therefore enjoys the properties of the latter given in No. 18.

**Theorem 7.** The classes \( O_G, O_{HD}, \) and \( O_{KD} \) are preserved under quasi-isometries.

26. By way of preparation we insert here, for the convenience of the reader, two elementary properties of nonnegative matrices. We recall that a matrix \( A \) is nonnegative (positive) if the form \( (Ax,x) \) is positive semidefinite (definite resp.).

Let \( A \) and \( B \) be two \( n \times n \) symmetric matrices. The inequality \( A \leq B \) \((A < B)\) means that \( B - A \) is a nonnegative (positive, resp.) matrix. The standard notations \( A^{-1}, A^t, \) and \(|A|\) are used for the inverse matrix, the transposed matrix, and the determinant of \( A \).

**Lemma 3.** Let \( A \geq 0 \) and \( B > 0 \). Then the inequality \( A \leq B \) implies that \(|A| \leq |B|\).

To see this take the orthogonal matrix \( P \) such that \( P^tBP \) is a diagonal matrix with elements \( \lambda_1, \lambda_2, \ldots, \lambda_n \), say. Since \( B > 0 \) implies \( \lambda_i > 0 \), \( i = 1, 2, \ldots, n \), we can consider the diagonal matrix \( C \) with elements \( \lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1} \). Let \( Q \) be the orthogonal matrix such that \( Q^t(PC)^tA(PC)Q \) is diagonal with elements \( \mu_1, \mu_2, \ldots, \mu_n \). Then \( A \leq B \) implies for \( S = PCQ \)

\[
S^tA S \leq S^tB S .
\]

In terms of the \( n \times n \) unit matrix \( E \) and the \( 1 \times n \) matrix \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \) we have

\[
S^tA S = \mu E , \quad S^tB S = E ,
\]

and therefore

\[
\mu E \leq E .
\]

This in turn implies that \( \mu_i \leq 1 \). Since \( A \geq 0 \), we obtain \( S^tA S = \mu E \geq 0 \), and thus \( \mu_i \geq 0 \). Therefore \( \prod_1^n \mu_i \leq 1 \), i.e., \(|\mu E| \leq |E|\). Substitution by (19) gives \(|A| \leq S|^2 \leq |B| \leq S|^2 \), and the assertion \(|A| \leq |B|\) follows.

27. For positive matrices we shall show:

**Lemma 4.** The inequality \( A \leq B \) with \( A > 0 \) and \( B > 0 \) implies that \( B^{-1} \leq A^{-1} \).
Using the notations of No. 26 we observe that $A > 0$ implies $S^t A S > 0$ and a fortiori $\mu_1 > 0$. This together with (20) gives $0 < \mu_1 \leq 1$ and consequently $1 \leq \mu_1^{-1}$. Therefore

$$E^{-1} \leq (\mu E)^{-1},$$

which in terms of (19) is

$$(S^t B S)^{-1} \leq (S^t A S)^{-1}.$$ 

Hence $S(S^t B S)^{-1} S^t \leq S(S^t A S)^{-1} S^t$, and this is equivalent to $B^{-1} \leq A^{-1}$.

28. We turn to the proof of Theorem 7. To show that quasi-isometries are Dirichlet mappings we are to compare Dirichlet integrals over $V$ and $\bar{V}$. To this end we compare volume elements in the present No., and gradients in No. 29.

Choose a set $E \subset V$ such that (i) meas $E = 0$, (ii) $T$ is totally differentiable in $V - E$, (iii) $T^{-1}$ is totally differentiable in $\bar{V} - T(E)$, and (iv) $J > 0$ in $V - E$. All considerations below will be in $V - E$ and $\bar{V} - T(E)$.

Let $\bar{x} = T(x)$. Then by (12),

$$K^{-1} ds \leq d\bar{s} \leq K ds,$$

or equivalently,

$$K^{-2} g_{ij} dx^i dx^j \leq \bar{g}_{ij} \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} dx^k dx^l \leq K^2 g_{ij} dx^i dx^j.$$ 

In terms of the matrix $M = (\partial \bar{x}^i/\partial x^j)$, with $i$ indicating the row, $j$ the column, this can be written

$$K^{-2} (g_{ij}) \leq M^t (\bar{g}_{ij}) M \leq K^2 (g_{ij}).$$

On taking the determinants we obtain by Lemma 3

$$K^{-2n} g \leq \bar{g}|M|^2 \leq K^{2n} g.$$ 

For volume elements $dV = g^{ij} dx^i \ldots dx^n$ and $d\bar{V} = \bar{g}^i J dx^1 \ldots dx^n$ this takes the form

$$K^{-n} dV \leq d\bar{V} \leq K^n dV.$$ 

29. To compare gradients on $V$ and $\bar{V}$ we obtain from Lemma 4 and (22)

$$K^{-2} (g^{ij}) \leq M^{-1} (\bar{g}^{ij})(M^t)^{-1} \leq K^2 (g^{ij}).$$

A left and right multiplication by $M$ and $M^t$ respectively gives

$$K^{-2} (\bar{g}^{ij}) \leq M (g^{ij}) M^t \leq K^2 (\bar{g}^{ij}).$$
First consider functions $f \in \mathcal{D}^\infty(\tilde{V})$. Then clearly $f \circ T \in \mathcal{D}(V)$. On applying the $n \times 1$ matrix $(\partial f/\partial \tilde{x}^i)$ on the right and its transpose on the left to (24) we obtain

$$K^{-2} \tilde{g}^{ij} \frac{\partial f}{\partial \tilde{x}^i} \frac{\partial f}{\partial \tilde{x}^j} \leq g^{kl} \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial \tilde{x}^j}{\partial x^l} \frac{\partial f}{\partial \tilde{x}^i} \frac{\partial f}{\partial \tilde{x}^j} \leq K^2 \tilde{g}^{ij} \frac{\partial f}{\partial \tilde{x}^i} \frac{\partial f}{\partial \tilde{x}^j}$$

a.e. in $V-E$. The expression in the middle is

$$g^{kl} \frac{\partial f}{\partial x^k} \frac{\partial f}{\partial x^l} = |\text{grad} f \circ T|^2,$$

and we obtain

(25) \hspace{1cm} \begin{align*} K^{-2} |\text{grad} f|^2 & \leq |\text{grad} f \circ T|^2 \leq K^2 |\text{grad} f|^2 \\ & \text{a.e. in } V-E. \end{align*}

On combining (23) and (25) we obtain the desired relations

(26) \hspace{1cm} \begin{align*} K_1^{-1} D(f) & \leq D(f \circ T) \leq K_1 D(f), \\ & \text{with } K_1 = K^{n+2}. \end{align*}

In the general case of $f \in \mathcal{D}(\tilde{V})$ we argue as in No. 21 to conclude that $f \circ T$ is in $\mathcal{D}(V)$ and satisfies (26). This completes the proof of Theorem 7.

REFERENCES


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