ON SUBGROUPS OF THE FIRST KIND

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In his fundamental work on discontinuous groups C. L. Siegel [1] calls a discrete subgroup $H$ of a locally compact group $G$ which satisfies the second axiom of countability a subgroup of the first kind if the following conditions are satisfied. There is a fundamental domain $F$ relative to $H$ of finite measure which has only a finite number of neighbours and which is normal. A translate $Fh$ of $F$ by an element $h$ in $H$ is called a neighbour of $F$ if $Fh \cap F$ is not empty so that the condition that $F$ has but finitely many neighbours means that $F^{-1}F \cap H$ is finite. That $F$ is normal means that the covering of $G$ by the family $\{Fh : h \in H\}$ is locally finite, that is, every point of $G$ has a neighbourhood which meets $Fh$ for only a finite number of $h \in H$. It is furthermore clear that if $F$ is normal, then there exists an open set $U$ such that $F \subset U \subset FH_0$ where $H_0$ consists of those $h$ in $H$ for which $Fh$ is a neighbour of $F$. On the other hand, the existence of such a set $U$ together with the finiteness of $H_0$ implies normality of $F$.

We are interested here in the fact, pointed out by Siegel (see [1, p. 683]), that if $G/H$ is compact, then $H$ is of the first kind. For the compactness of $G/H$ implies the existence of a fundamental domain $F$ having compact closure and the conditions that $H$ be of the first kind are clearly met by such an $F$.

We observe that the compactness of $F$ implies rather more in respect of normality namely that an open set $U$ satisfying the condition $F \subset U \subset FH_0$ is realizable in the form $FV$ where $V$ is an open neighbourhood of the identity, $e$, having compact closure. For the assumptions on $G$ provide that there exists a sequence $(V_i)$ of open neighbourhoods of $e$ in which $V_{i+1} \supset V_i$, $i = 1, 2, \ldots$, is compact and $\cap V_i = e$. Since $F^{-1}FV_i$ is compact and $H$ is discrete, it follows that $F^{-1}FV_i \cap H$ is a finite set and, choosing $i$ sufficiently large, $F^{-1}FV_i \cap H = H_0$. In regard to normality, we have, in these circumstances that for any $x$ in $G$ there is a neighbourhood $V$ of $e$ which is independent of $x$ such that $Vx$ is covered by a

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finite number of translates $Fh$, $h \in H$, this number being at most the number of elements in $H_0$.

The above observation leads us to make the following definition: A discrete subgroup $H$ is uniformly of the first kind when there exists a fundamental domain $F$ relative to $H$ of finite measure for which $H_0 = \overline{F}^{-1}F \cap H$ is finite and there exists an open neighbourhood $V$ of $e$ such that $\overline{F} \subset \overline{F}V \subset FH_0$. Stated in these terms what we have observed above is that if $G/H$ is compact, then $H$ is uniformly of the first kind. What we are now going to show is that the converse assertion holds. It is a special case of the following

**Theorem.** Let $G$ be a locally compact $\sigma$-compact group with left invariant measure $m$. Let $A$ be a subset of $G$ of finite measure which is such that there exists a covering $\{A\lambda: \lambda \in \Lambda\}$ of $G$ in which, for some compact neighbourhood $V$ of $e$, $AV \cap A\lambda \neq \emptyset$ for only a finite subset of $\Lambda$. Then $\overline{A}$ is compact.

**Proof.** Let $A_0 = A \cap A^{-1}AV$ where $V$ is as given. Denoting $AA_0$ by $\overline{A}$, we have that $m(\overline{A}) \leq |A_0|m(A)$, where $|A_0|$ is the number of elements in $A_0$. It follows that $m(\overline{A})$ is finite.

The hypotheses on $G$ provide that $G$ is the countable union of compact sets, say $G = \bigcup C_i$, in which we may assume that $C_i \subset C_{i+1}$ for $i = 1, 2, \ldots$.

Suppose that $\overline{A}$ is not compact. Then for each $i$, $\overline{A} - C_iV^{-1}$ is not empty since $C_iV^{-1}$ is compact. Let $x$ be any point in $\overline{A} - C_iV^{-1}$, then $xV \cap C_i = \emptyset$. But $xV \subset \overline{A}$ so that $\overline{A}$ has a compact subset which by the left invariance of $m$ has measure $m(V)$ outside $C_i$ for each $i$. Since $m(V) > 0$, this contradicts the fact that $m(\overline{A})$ is finite. It follows that $\overline{A}$ is compact.

**Reference**


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