ASYMPTOTES OF CONVEX BODIES

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For two subsets $A$ and $X$ of Euclidean $d$-space $E^d$, let

$$\delta(A, X) = \inf \{ \|a - x\| : a \in A, \ x \in X \} .$$

The set $A$ is a $j$-asymptote of $X$ provided that $A$ is a $j$-dimensional flat with

$$A \cap X = \emptyset \quad \text{and} \quad \delta(A, X) = 0 .$$

Thus $X$ admits a 0-asymptote if and only if $X$ fails to be closed, and $X$ admits a $j$-asymptote if and only if $X$'s orthogonal projection on some $(d - j)$-dimensional flat in $E^d$ fails to be closed. For each convex body $C$ in $E^d$ (closed convex set with nonempty interior) let $\alpha C$ denote the set of all integers $j$ such that $C$ admits a $j$-asymptote. If $\alpha C \neq \emptyset$ then (as noted in [2]) $\alpha C = \{ j : 1 \leq j \leq d - 1 \}$ when $C$ has no boundary ray and $\alpha C = \{ j : 1 \leq j \leq d - 2 \}$ when $C$ is a cone. Here we settle a problem raised in [2] by showing that every set of integers between 1 and $d - 1$ can be realized as the set $\alpha C$ for suitably constructed convex bodies $C$ in $E^d$. The construction is adapted from [1].

**Theorem.** For each set $J = \{ j : 1 \leq j \leq d - 1 \}$ there is a convex body $C$ in $E^d$ such that $C$ contains no line and $\alpha C = J$. 

**Proof.** The assertion being obvious for $d \leq 2$, we proceed by induction on $d$. Suppose the assertion known for $E^d$ and consider a set $J$ of integers between 1 and $d$. We want to produce a convex body $K$ in $E^{d+1}$ such that $K$ contains no line and $\alpha K = J$. Let $C$ be a convex body in $E^d$ such that $C$ contains no line and

$$\alpha C = \{ h : 1 \leq h \leq d - 1 \ \text{and} \ h + 1 \in J \} .$$

Choose an extreme point $p$ of $C$ (possible since $C$ contains no line) and let

$$X = C \quad \text{if} \ 1 \notin J, \quad X = C \sim \{ p \} \quad \text{if} \ 1 \in J .$$

Received November 19, 1966.

This research was supported in part by the National Science Foundation, U.S.A. (NSF-GP-3579).
In each case $X$ is a convex $F_{a}$ set and hence is the union of an increasing sequence $Y_{1} = Y_{2} = \ldots$ of compact convex sets such that $\|y\| \leq i$ for all $y \in Y_{i}$. Let $E^{d}$ be embedded as usual in $E^{d+1}$, so that $E^{d+1} = E^{d} \oplus Rz$ for a line $Rz$ orthogonal to $E^{d}$. Finally, let

$$K = \text{con} \bigcup_{i} (Y_{i} \oplus i^{2}z),$$

so that $X$ is the orthogonal projection of $K$ on $E^{d}$. On p. 101 of [1] it is proved that $K$ is closed, whence of course $K$ is a convex body containing no line. Plainly

$$\alpha K = \{h + 1 : h \in \alpha X\},$$

for $A \oplus Rz$ is an asymptote of $K$ in $E^{d+1}$ whenever $A$ is an asymptote of $X$ in $E^{d}$. From the choice of $C$ and from the care in defining $X$ when $1 \in J$ it follows that

$$\{h + 1 : h \in \alpha X\} = J.$$

Thus to complete the proof it suffices to show that $\alpha K \subset J$, or equivalently that $j \in \alpha K$ implies $j - 1 \in \alpha X$. Note first that no asymptote of $K$ is parallel to $E^{d}$, for $K$ is closed and lies in paraboloidal region

$$(*) \quad Q = \{y \oplus rz : y \in E^{d}, \ r \geq 0, \ \|y\| \leq r^{1}\}$$

whose intersection with any translate of $E^{d}$ is compact.

Now consider an arbitrary $j$-asymptote $A$ of $K$. For each $r \in R$ let $A_{r}$ denote the $(j - 1)$-flat $A \cap (E^{d} \oplus rz)$ and let $P_{r}$ denote the orthogonal projection of $A_{r}$ on $E^{d}$. Note that $A_{r} = A_{0} + r(A_{1} - A_{0})$, whence $P_{r} = A_{0} + r(P_{1} - A_{0})$ and

$$(\ast) \quad \delta(P_{r}, A_{0}) = r \delta(P_{1}, A_{0}).$$

If $P_{1} = A_{0}$ then $A = A_{0} \oplus Rz$ and $A_{0}$ is plainly a $(j - 1)$-asymptote of $X$. If $P_{1} + A_{0}$ then $\delta(P_{1}, A_{0}) > 0$ and it follows from $(\ast)$ and $(\ast)$ that

$$\delta(A_{r}, Q \cap (E^{d} \oplus [0, 4r]z) \geq \delta(P_{r}, \{y \in E^{d} : \|y\| \leq 2r^{1}\})$$

$$\quad \geq \delta(P_{1}, A_{0})r - 2r^{1} - \delta(A_{0}, \{0\}),$$

whence $\lim_{r \to \infty} \delta(A_{r}, Q) = \infty$ and $A$ is not an asymptote of $K$. This completes the proof.

REFERENCES