RINGS WITH PRIMARY IDEALS AS MAXIMAL IDEALS

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1. Introduction.

If every primary ideal is a maximal ideal in a commutative ring \( R \) with identity, then we say that \( R \) is a \( P \)-ring. Evidently in \( P \)-rings primary, prime and maximal ideals coincide. Commutative von Neumann regular rings and in particular a direct sum of finite number of fields are \( P \)-rings. In general it can be shown that a \( P \)-ring is a subdirect sum of fields (trivial regular rings) and furthermore if a \( P \)-ring is Noetherian, it is a finite direct sum of fields. This characterization enables us to prove that rings mentioned in section 3 are \( P \)-rings and hence direct sums of a finite number of fields.

2. \( P \)-rings.

**Theorem.** If \( R \) is a \( P \)-ring then \( R \) is a subdirect sum of fields. In addition, if \( R \) is Noetherian, then \( R \) is a finite direct sum of fields.

**Proof.** Since, in \( P \)-rings, prime and maximal ideals coincide, the intersection of all primary ideals is \((0)\) by virtue of a result due to Krull mentioned in [4, p. 492]. Hence the intersection of all maximal ideals, i.e. the Jacobson radical of \( R \), is \((0)\). This implies that \( R \) is a subdirect sum of fields.

If \( R \) is a Noetherian \( P \)-ring, then \( R \) satisfies the descending chain condition since prime ideals are maximal. But \( R \) has zero Jacobson radical. Hence \( R \) is a finite direct sum of fields.

If the defining property of a \( P \)-ring is replaced by the property that every proper-primary ideal is maximal, then the above theorem need not be true, as the example "\( R = \) the integers modulo 4", shows.

There exist \( P \)-rings which are not Noetherian. For example, the complete direct product of infinitely many copies of a (commutative) field is a non-Noetherian \( P \)-ring.

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3. Applications.

We shall first establish the main consequence of the theorem in section 2, in order to deduce theorem 3.2 of this section.

**Theorem (3.1).** Let $R$ be a commutative ring with identity in which every maximal ideal is generated by an idempotent. Then $R$ is a direct sum of a finite number of fields.

**Proof.** We begin by proving that $R$ is a $P$-ring. Let $A$ be any primary ideal in $R$ which is not maximal. Then $A$ is included in a proper maximal ideal, say $eR$, where $e$ is an idempotent different from 0 and 1. Evidently $e \not\in A$. But $e(1-e)=0$. This implies $(1-e)^n \in A$ for some positive integer $n$, since $A$ is a primary ideal. Hence $(1-e) \in eR$. Thus $1 \in eR$, a contradiction.

Since $R$ is a $P$-ring, prime ideals are maximal. Therefore every prime ideal is finitely generated (principal) by virtue of the hypothesis. This implies that $R$ is Noetherian by the application of Cohen’s result [2, Theorem 2]. Hence it follows from 2.1 that $R$ is a direct sum of a finite number of fields.

**Theorem (3.2).** Let $R$ be a commutative ring with identity. Then the following are equivalent:

1. $R$ is a finite direct sum of fields.
2. Every maximal ideal is generated by an idempotent.
3. Every maximal ideal is a direct summand of $R$.
4. Every maximal ideal is $R$-projective as a right $R$-module and is principally generated by a zero-divisor.
5. Every proper maximal ideal is $R$-injective as a right $R$-module.
6. $R$ has no nilpotents and every proper maximal ideal has a non-zero annihilator.

**Proof.** Since (1) implies everyone of the other stated conditions, we have by 3.1, $(1) \iff (2) \iff (3)$. Hence it suffices to show that each one of the conditions (4), (5) and (6) separately imply (2) or (3).

(4) $\implies$ (2): Let $M$ be an arbitrary maximal ideal and let $M = xR$, $x$ being a zero-divisor. Then $x^r$, the annihilator of $x$ is non-zero. Consider the exact sequence of $R$-modules,

$$0 \longrightarrow x^r \overset{i}{\longrightarrow} R \overset{j}{\longrightarrow} xR \longrightarrow 0,$$

where "$i$" is an inclusion mapping and $j: a \to xa$, $a \in R$. Since $xR$ is projective, the above exact sequence splits. Then $x^r$ is a direct summand
of \( R \). This implies \( x^r = eR, e \) being an idempotent \( \neq 0 \) and 1, since \( R \) has an identity. Now \( 0 = xe \). Therefore

\[ x = x(1 - e) \quad \text{and} \quad xR \subseteq (1 - e)R. \]

This in turn implies that \( xR = (1 - e)R \) since \( xR \) is a maximal ideal. Thus every maximal ideal is generated by an idempotent.

(5) \( \Rightarrow \) (2): Let \( M \) be a proper maximal ideal. Since \( M \) is \( R \)-injective by hypothesis, \( M \) is a direct summand of \( R \) [1, prop. 3.4] and hence (2) follows.

(6) \( \Rightarrow \) (3): If \( M \) is a proper maximal ideal and if \( M^* \) is its non-zero annihilator, then \( M \cap M^* = 0 \). For, take any \( x \in M \cap M^* \). Then \( x^2 = 0 \), hence \( x = 0 \). This implies \( R = M \oplus M^* \) and hence (3).

Remark (3.3). Comparing the condition (4) of theorem 3.2, with an important theorem of Kaplansky [3, theorem 2.3] we observe that the projective nature of all maximal ideals does all that an ascending chain condition can, and thus make the ring a principal ideal ring. In addition we obtained a nice structure of the ring.

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REFERENCES


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