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ON THE CONVERGENCE PRINCIPLE OF B. M. KLOSS

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1. Introduction.

Convergence principles for probability measures on locally compact topological groups G are concerned with a proper "norming" of convolution sequences of probability measures on G such that the "normed" sequence always converges to a limit measure.

The notion of convergence principle dates back to a paper by B. M. Kloss in 1959 [4]. Kloss showed that on first countable compact groups a convergence principle holds. The following note serves two purposes: First of all it replaces the (incomplete) proof of Kloss by a new one relying on a general theorem of Csiszár [1]. Secondly a converse of Kloss' theorem is proved which yields a characterization of those groups where the convergence principle is valid.

The author owes gratitude to Dr. I. Csiszár for having put to his disposal a manuscript which contains a key Lemma useful in the first part of the proof of the theorem and for a helpful comment.

2. Preliminaries.

Let G be a locally compact (Hausdorff) group with Haar measure ω. By $\mathscr{K} := \mathscr{K}(G)$ we denote the set of all real valued continuous functions on G having compact support. Let $\mathscr{C}^\infty := \mathscr{C}^\infty(G)$ stand for the set of all real valued continuous functions on G vanishing at infinity. Let $\mathscr{M} := \mathscr{M}(G)$ be the set of all Radon probability measures on G (i.e. the set of all regular Borel probability measures on the Borel σ-algebra $\mathfrak{B} := \mathfrak{B}(G)$ of G). In $\mathscr{M}$ one can introduce the vague topology $\mathscr{T}_v$ such that a sequence $(\mu_n)_{n\geqq 1}$ in $\mathscr{M}$ converges to $\mu \in \mathscr{M}$ in the sense of $\mathscr{T}_v$ if and only if the sequence $(\mu_n(f))_{n\geqq 1}$ converges to $\mu(f)$ for all $f \in \mathscr{K}$, as $n \to \infty$. We shall use the abbreviation $\mu_n \to \mu$ or $\lim_{n\to\infty}\mu_n = \mu$ for convergence in $\mathscr{T}_v$.

We also introduce in $\mathscr{M}$ a multiplication, viz., convolution of probability measures (denoted by $*$). It is known that the mapping $(\mu,\nu) \to \mu*\nu$ of $\mathscr{M} \times \mathscr{M}$ into $\mathscr{M}$ is $\mathscr{T}_v$-continuous. Hence $\mathscr{M}$ becomes a topological semi-group. By ε_x we denote the Dirac measure in x ($x \in G$). Given

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$\mu,\nu \in \mathcal{M}$, we write $\mu \sim \nu$ if there exists an $x \in G$ such that $\mu = \nu * \varepsilon_x$. There is an involution in $\mathcal{M}$ defined by $\mu^{\sim}(B) = \mu(B^{-1})$ for all $B \in \mathfrak{B}$ ($\mu \in \mathcal{M}$).

Finally we need to mention two facts from the analysis of $\mathcal{M}$ which pertain to the tools of this paper.

(i) Support formula: Let $\mu,\nu \in \mathcal{M}$ and let S_μ, S_ν be the corresponding supports. Then $S_{\mu*\nu} = \overline{S_\mu S_\nu}$ (the bar meaning the closure operation).

(ii) Characterization of idempotents in $\mathcal{M}$: The idempotents in $\mathcal{M}$ are exactly the Haar measures ω_H for compact subgroups H of G.

The latter result has been proved by the author in [2].

3. The characterization theorem.

Let $(\mu_n)_{n \geqq 1}$ be a sequence in $\mathcal{M}$. Denote by ν_k the k-fold convolution $\mu_1 * \ldots * \mu_k$ of elements of $(\mu_n)_{n \geqq 1}$. We next consider the sequence $(\nu_k)_{k \geqq 1}$ in $\mathcal{M}$.

Definition. We say that *for G the convergence principle* (CP) *holds*, if for any sequence $(\mu_n)_{n \geqq 1}$ in $\mathcal{M}$ there exists a sequence $(x_n)_{n \geqq 1}$ in G such that the (right normed) sequence $(\nu_k * \varepsilon_{x_k})_{k \geqq 1}$ converges.

Remark. It is clear that, in the case of a compact group G, any sequence $(\mu_n)_{n \geqq 1}$ in $\mathcal{M}$ converging to μ has the following property: For each sequence $(x_n)_{n \geqq 1}$ in G the sequence $(\mu_n * \varepsilon_{x_n})_{n \geqq 1}$, provided it converges, will converge to a limit measure λ such that $\lambda \sim \mu$.

We first state a theorem of Csiszár proved for second countable locally compact groups and draw a consequence.

Theorem [1]. *Let $(\mu_n)_{n \geqq 1}$ be a sequence in $\mathcal{M}$. Define a second sequence $(\mu_{k,l})_{0 \leqq k < l}$* by

$$\mu_{k,l} := \mu_{k+1} * \mu_{k+2} * \ldots * \mu_{k+l}$$

and let $\mu \in \mathcal{M}$ be an accumulation point of the sequence $(\mu_{0,l})_{l \geqq 1}$. Then

(i) *There exists a subsequence $(n_j)_{j \geqq 1}$ in* N [1]) *such that $\hat{\mu}_k := \lim_{j \to \infty} \mu_{k,n_j}$ ($k \geqq 0$; $\hat{\mu}_0 := \mu$) and $\mu_\infty := \lim_{j \to \infty} \hat{\mu}_{n_j}$ exist.*

(ii) *μ_∞ is idempotent and $\hat{\mu}_k = \hat{\mu}_k * \mu_\infty$ for all $k \geqq 0$.*

(iii) *If $\mu, \mu' \in \mathcal{M}$ are two accumulation points of the sequence $(\mu_{0,l})_{l \geqq 1}$ and if $(n_i)_{i \geqq 1}$ is a subsequence in* N *such that $\hat{\mu}_k' := \lim_{i \to \infty} \mu_{k,n_i}$ ($k \geqq 0$; $\hat{\mu}_0' := \mu'$) and $\mu_\infty' := \lim_{i \to \infty} \hat{\mu}_{n_i}$ exist, then for arbitrary accumulation points ν', ν'' of the sequences $(\hat{\mu}_{n_i})_{i \geqq 1}$, $(\hat{\mu}'_{n_i})_{i \geqq 1}$ resp. we get $\mu' = \mu * \nu'$ and $\nu' * \nu'' = \mu_\infty$.*

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[1]) N denotes the set of all natural numbers (positive integers).

COROLLARY. *Let* $(\mu_n)_{n\geqq 1}$ *be a sequence in* $\mathscr{M}$ *and* $(\lambda_k)_{k\geqq 1}$ *the sequence of finite convolution products*

$$\lambda_k := \mu_{0,k} := \mu_1 * \ldots * \mu_k .$$

Then, if $(\lambda_k)_{k\geqq 1}$ *has accumulation points* λ, λ' *in* $\mathscr{M}$, *one has* $\lambda' \sim \lambda$.

PROOF OF COROLLARY. Let λ, λ' be two accumulation points of $(\lambda_k)_{k\geqq 1}$ which are elements of $\mathscr{M}$. By (iii) we have $\lambda' = \lambda * \nu'$ and $\nu' * \nu'' = \mu_\infty$ which by (ii) and the characterization of idempotents equals ω_H, where H is a compact subgroup of G. The support formula yields $S_{\nu'} \subset Hx^{-1}$ for all $x \in S_{\nu''}$. Putting $x_0 = x^{-1}$ we get $\omega_H * \nu' = \omega_H * \varepsilon_{x_0}$ and finally

$$\lambda' = \lambda * \nu' = \lambda * \omega_H * \nu' = \lambda * \varepsilon_{x_0},$$

using (ii) with $k = 0$.

THEOREM. (Characterization of (CP).) (i) *If* G *is compact and first countable, then* (CP) *holds.*

(ii) *If* (CP) *holds, then* G *is compact.*

REMARKS. For first countable groups, (CP) is equivalent to the compactness of G.—Statement (i) of the theorem is due to Kloss [4, p. 254].

PROOF OF (i). Let G be compact and first countable. Let $(\mu_n)_{n\geqq 1}$ be any sequence in $\mathscr{M}$. Denote by $\mathscr{H} := \{\nu_t : t \in T\}$ the set of all accumulation points of the sequence $(\nu_k)_{k\geqq 1}$, where

$$\nu_k := \mu_1 * \ldots * \mu_k .$$

Let ν be a fixed element in $\mathscr{H}$. By the above corollary we know that for each $\nu_t \in \mathscr{H}$ we have $\nu_t \sim \nu$, that is, to each $\nu_t \in \mathscr{H}$ there exists an $x_t \in G$ such that $\nu_t = \nu * \varepsilon_{x_t}$.

Since G is assumed to be first countable, we can exhibit a decreasing sequence $(U_n)_{n\geqq 1}$ of neighborhoods of ν as a base. The subset $\mathscr{H}$ of $\mathscr{M}$ is $\mathscr{T}_v$-compact and

$$\mathscr{H} \subset \bigcup_{t \in T} U_n * \varepsilon_{x_t} .$$

Thus for each $n \geqq 1$ there exists a finite subset $\{\varepsilon_{x_{n_1}}, \ldots, \varepsilon_{x_{n_r}}\}$ of the set $\{\varepsilon_{x_t} : t \in T\}$ such that

$$\mathscr{H} \subset \bigcup_{i=1}^{r} U_n * \varepsilon_{x_{n_i}} .$$

By definition of $\mathscr{H}$ there is an $N = N(n) \in \mathbf{N}$ such that for all $n \geqq N$

$$\nu_k \in \bigcup_{i=1}^{r} U_n * \varepsilon_{x_{n_i}} .$$

For $n \in [N(n), N(n+1)[$ we choose $\varepsilon_{x_{n_i}}$ such that $\nu_k \in U_n * \varepsilon_{x_{n_i}}$. Putting

$\varepsilon_{y_k} := \varepsilon_{x_{ni}^{-1}-1}$ we get $\nu_k * \varepsilon_{y_k} \in U_n$ for all $k \geqq N(n)$, hence $(\nu_k * \varepsilon_{y_k})_{k \geqq 1}$ converges to ν.

Before we go into the proof of (ii) we first prove the following

LEMMA. *Let G be a locally compact noncompact group. Then there exists on G a function in $\mathscr{C}^\infty$ with noncompact support.*

PROOF. Let U be a relatively compact open neighborhood of the identity $e \in G$. Then the family $\mathfrak{U} = \{xU : x \in G\}$ is a covering of G by open sets, which does not admit a finite subcovering. In fact, if $\mathfrak{U}$ would admit a finite subcovering of G, G would be a finite union of compact sets and therefore itself compact, which contradicts the assumption. Hence there exists an infinite sequence $(x_n)_{n \geqq 1}$ in G such that

$$x_n \in \complement \bigcup_{k=1}^{n-1} x_k U \quad \text{for} \quad n \geqq 2 \, .$$

Let, now, V be a symmetric neighborhood of e with $V^2 \subset U$. The elements of the sequence $(x_n V)_{n \geqq 1}$ are pairwise disjoint. On G there exists a continuous function f such that $0 \leqq f \leqq 1$, $f(e) = 1$ and $f(x) = 0$ for all $x \in \complement V$. It then follows that the function g on G defined by

$$g(x) := \sum_{n=1}^{\infty} 2^{-n} f(x_n{}^{-1} x)$$

is continuous and vanishes at infinity (that is, $g \in \mathscr{C}^\infty$), but its support S_g is noncompact by construction.

REMARK. The above construction shows that for noncompact locally compact groups, $\mathscr{C}^\infty$ is never contained in $\mathscr{K}$ (see [3, p. 152]).

PROOF of (ii). We assume that G is a noncompact locally compact group and that (CP) is valid. Our aim will be to derive from that a contradiction. The above Lemma guarantees the existence of a Borel measurable real valued function $f \geqq 0$ with noncompact support S_f such that

$$\int_G f \, d\omega = 1 \, .$$

The measure $\mu \in \mathscr{M}$ defined by

$$\mu(B) = \int_B f \, d\omega \quad \text{for all} \quad B \in \mathfrak{B}$$

has compact support S_μ if and only if S_f is compact. Hence we have arrived at the existence of a measure $\mu \in \mathscr{M}$ with noncompact S_μ. Clearly μ can also be chosen symmetric (that is, $\mu^\sim = \mu$).

We apply (CP) to this very measure μ. It follows that there exists a sequence $(x_n)_{n \geqq 1}$ in G such that

$$\lim_{n\to\infty} \nu_n := \lim_{n\to\infty} \mu^{*n} * \varepsilon_{x_n} \quad \text{exists}$$

(and equals ν, say). It is also clear that

$$\lim_{n\to\infty} \nu_n^\sim \quad \text{exists}$$

(and equals $\nu^\sim$).

By the continuity of the mapping $(\mu,\nu) \to \mu * \nu$ from $\mathscr{M} \times \mathscr{M}$ into $\mathscr{M}$ in the sense of $\mathscr{T}_v$ we conclude that $\lim_{n\to\infty} \nu_n * \nu_n^\sim = \nu * \nu^\sim$, and hence

$$\lim_{n\to\infty} \nu_n * \nu_n^\sim = \lim_{n\to\infty} (\mu^{*n} * \varepsilon_{x_n}) * (\varepsilon_{x_n^{-1}} * \mu^{\sim *n}) = \lim_{n\to\infty} \mu^{*2n},$$

which we define to be $\lambda\,(= \nu * \nu^\sim)$.

On the other hand we have $\lim_{n\to\infty} \nu_n * \nu_n^\sim = \lambda^{*2}$. Thus $\lambda^{*2} = \lambda$ and λ is idempotent in $\mathscr{M}$. By the characterization of idempotents in (ii) of section 2 there exists a compact subgroup H of G such that $\lambda = \omega_H$. Note that $S_\lambda = H$. Since

$$\lim_{n\to\infty} \mu^{*2(n+1)} = \lim_{n\to\infty} \mu^{*2} * \mu^{*2n} = \left(\lim_{n\to\infty} \mu^{*2}\right) * \left(\lim_{n\to\infty} \mu^{*2n}\right) = \mu^{*2} * \lambda,$$

we also get $\lambda = \mu^{*2} * \lambda$.

Since μ is the measure in $\mathscr{M}$ that we started off with, the support formula in (i) of section 2 yields that $S_{\mu^{*2}}$ is noncompact and hence so is $S_{\mu^{*2} * \lambda}$. But then S_λ is concluded to be noncompact which is the desired contradiction, since $S_\lambda = H$, H being a compact subgroup of G.

4. Comments.

(1) In the case of second countability the characterization theorem also follows from Theorem 3.1 in [1], but the argument there does not seem to extend to the non-second-countability situation.

(2) The argument in part (ii) of the proof, however, can be simplified, if G is assumed to be second countable, as is evident from the following fact: On any second countable noncompact locally compact space X there exists a purely atomic measure $\mu \in \mathscr{M}$ with noncompact S_μ. In fact, since G is noncompact, there is a countable set $\{x_n : n \geqq 1\}$ in G without accumulation points. One defines μ by $\mu(\{x_n\}) = 2^{-n}$ for $n \geqq 1$. Then μ has the required property.

(3) A different version of (CP) appears in a paper by Tortrat [5]: G is assumed to be a completely regular group, $\mathscr{M}$ stands for the set of all tight normed positive functionals on the set of all bounded continuous functions on G, and convergence in the sense of $\mathscr{T}_v$ is replaced by uniform tightness of families of functionals.

NOTE. After the acceptance of this paper for publication, Professor A. Tortrat has informed the author that further studies of the above version of (CP) have been published by him in a recent paper [*Lois tendues et convolutions dénombrables dans un groupe topologique* X, Ann.Inst. H. Poincaré (B) 2 (1966), 279–298]. Furthermore it has been pointed out that an idea similar to the one used in the proof of statement (ii) of the theorem is contained in a paper by B. M. Kloss [*Topology in a group and convergence of distributions*, Theor. Probability Appl. 9 (1964), 111–114].

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