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BOUNDARY VALUES FOR HOMOMORPHISMS OF COMPACT CONVEX SETS

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The aim of the present paper is to characterize those maps of the extreme boundary which are extendable to continuous affine maps (homomorphisms) of the whole set, and to apply the results to the Dirichlet problem of the extreme boundary (Choquet boundary).

A map φ of a compact convex subset K_1 of a locally convex (Hausdorff) space E_1 into a compact convex subset K_2 of a locally convex (Hausdorff) space E_2 is said to be a *homomorphism* of K_1 into K_2 if it is continuous and *affine* in the sense that

$$(1) \qquad \varphi(\lambda x+(1-\lambda)x) = \lambda\varphi(x)+(1-\lambda)\varphi(x)$$

whenever $x,y \in K_1$ and $0 \leqq \lambda \leqq 1$. If in addition φ is $1-1$, then φ is said to be an *isomorphism* of K_1 into K_2. If there exists an isomorphism of K_1 onto K_2, then K_1 and K_2 are said to be *isomorphic.*

For every Borel map φ of a Borel subset A of a compact space K_1 into a compact space K_2, there exists an *associated map* $\hat{\varphi}$ of the set $\mathfrak{M}(A)$ of (regular Borel) measures concentrated on A into the set $\mathfrak{M}(K_2)$ of all measures on K_2. Explicitly $\hat{\varphi}$ is determined by the formula

$$(2) \qquad \hat{\varphi}\mu(E) = \mu(\varphi^{-1}(E)) ,$$

where E denotes a Borel subset of K_2, or by the equivalent formula

$$(3) \qquad \int f \circ \varphi \, d\mu = \int f \, d\hat{\varphi}\mu ,$$

where f denotes a bounded Borel function on K_2.

For every bounded real valued function f defined on a subset A containing the extreme boundary $\partial_e K$ of a compact convex set K in a locally convex space, there exist *upper* and *lower envelopes* $\bar{f}$ and $\underline{f}$. Specifically, $\bar{f}$ is the least upper semi-continuous concave function on K which majorizes f on A, and $\underline{f}$ is the greatest lower semi-continuous convex function on K which minorizes f on A (cf. e.g. [8, p. 140]).

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To every continuous real valued function f on a convex compact set K in a locally convex space, there is associated a *boundary set*

(4) $$B_f = \{x \mid x \in K, f(x) = \bar{f}(x)\} .$$

By a theorem of M. Hervé [10]:

(5) $$\partial_e K = \bigcap \{B_f \mid f \in \mathscr{C}(K)\} .$$

In fact, the formula (5) remains valid if $\mathscr{C}(K)$ is replaced by the smaller class of continuous and convex functions.

In the metrizable case there eixts an $f \in \mathscr{C}(K)$ such that $\partial_e K = B_f$ [10]. Hence in this case $\partial_e K$ is a G_δ-subset of K.

A measure μ on a compact convex set K in a locally convex space is said to be a *boundary measure* if it vanishes off every boundary set B_f, $f \in \mathscr{C}(K)$. In the metrizable case μ is a boundary measure if and only if it vanishes off the extreme boundary, i.e.

(6) $$|\mu|(K \setminus \partial_e K) = 0 .$$

In the general case a boundary measure μ vanishes off every Baire set containing $\partial_e K$, but this property alone does not characterize boundary measures (cf. [4], [11, Ch. 4]).

By the Choquet–Bishop–de Leeuw Theorem [6], [4] (cf. also [8], [11]), every point x of K is the *barycenter* (resultant) of some positive normalized boundary measure μ. In symbols

(7) $$x = \int t \, d\mu(t) \qquad \text{(weak integral)} .$$

The set of all positive normalized boundary measures with barycenter x will be denoted by $\mathfrak{M}_x^+$.

A (signed) measure μ on a convex compact set K in a locally convex space is said to be a (generalized) *affine dependence* if it has total mass zero and has resultant in the origin. An affine dependence which is at the same time a boundary measure, is said to be an *affine dependence on* $\partial_e K$ [1, p. 98]. The linear space of all affine dependences on K is denoted by $\mathfrak{N}(K)$, and the linear space of all affine dependences on $\partial_e K$ is denoted by $\mathfrak{N}(\partial_e K)$. If the extreme boundary is affinely independent, i.e. if $\mathfrak{N}(\partial_e K) = (0)$, then K is said to be a (Choquet-) *simplex*. Clearly K is a simplex if and only if $\mathfrak{M}_x^+$ has a unique member μ_x for every $x \in K$.

If A is some Borel set containing the extreme boundary $\partial_e K$ of a compact convex set K in a locally convex space, if f is a bounded Borel function on A, and if μ is a positive normalized measure concentrated on A with barycenter x, then by the definition of envelopes (cf. e.g. [8, p. 140])

$$(8) \qquad \underline{f}(x) \leqq \int f\,d\mu \leqq \bar{f}(x)\,.$$

If f is a continuous and convex function on K (i.e. $A=K$), then we have the following sharper result (cf. e.g. [8, p. 141]):

$$(9) \qquad \bar{f}(x) = \sup\left\{\int f\,d\mu \mid \mu \in \mathfrak{M}_x^{\ +}\right\} \qquad (\text{dually for } \underline{f})\,.$$

In the sequel we shall need the following simple consequence of Hervé's result (5):

LEMMA. *If f is a bounded* u.s.c. *function defined on the extreme boundary $\partial_e K$ of a compact convex set K in a locally convex space, then $f(x)=\bar{f}(x)$ for every $x \in \partial_e K$. Clearly, the dual statement, $\underline{f}(x)=f(x)$, is valid if f is* l.s.c.

PROOF. Assume $f(x) \geqq \alpha$ for all $x \in \partial_e K$, and define

$$f_1(x) = \begin{cases} \limsup\{f(y) \mid y \in \partial_e K,\, y \to x\}, & \text{if} \quad x \in \overline{\partial_e K} \\ \alpha, & \text{if} \quad x \in K \diagdown \overline{\partial_e K}\,. \end{cases}$$

Clearly f_1 is an u.s.c. extension of f from $\partial_e K$ to K. Let $\{g_\alpha\}$ be a descending net of continuous functions on K which converges pointwise to f_1. Clearly, the net $\{\bar{g}_\alpha\}$ is descending as well, and the pointwise limit $k=\lim_\alpha \bar{g}_\alpha$ is an u.s.c. concave majorant of f_1. In particular $k \geqq \bar{f}$. By Hervé's theorem, g_α and $\bar{g}_\alpha$ coincide on $\partial_e K$. Hence for every $x \in \partial_e K$

$$\bar{f}(x) \leqq k(x) = \lim_\alpha \bar{g}_\alpha(x) = \lim_\alpha g_\alpha(x) = f_1(x) = f(x)\,.$$

This completes the proof.

THEOREM. *Let K_i be a compact convex set in a locally convex space E_i for $i=1,2$, and assume K_1 metrizable. A continuous map φ of the extreme boundary $\partial_e K_1$ into K_2 can be extended to a homomorphism of K_1 into K_2 if and only if the following two requirements are satisfied:*

(i) $\hat{\varphi}\nu \in \mathfrak{N}(K_2)$ *for all* $\nu \in \mathfrak{N}(\partial_e K_1)$,
(ii) $\overline{f \circ \varphi}$ *and* $\underline{f \circ \varphi}$ *are continuous on* $\overline{\partial_e K_1}$ *for all* $f \in E_2^{\,*}$.

If $\partial_e K_1$ is closed, then (ii) *is autcmatically satisfied, and the conclusion holds without metrizability.*

PROOF. 1) Assume (i), (ii). For every $x \in K_1$, chose an arbitrary member μ of $\mathfrak{M}_x^{\ +}$. Now μ is concentrated on the G_δ-subset $\partial_e K_1$ of K_1, which is also the domain of definition of φ. Hence $\hat{\varphi}\mu$ is well defined. The barycenter of $\hat{\varphi}\mu$ does not depend on the particular choice of μ; for if μ' is another member of $\mathfrak{M}_x^{\ +}$, then

$$\mu - \mu' \in \mathfrak{N}(\partial_e K_1)\,,$$

and so by (ii)

$$\hat{\varphi}\mu - \hat{\varphi}\mu' \in \mathfrak{N}(K_2)\,,$$

and this means that $\hat{\varphi}\mu$ and $\hat{\varphi}\mu'$ have common barycenter. Thus there exists a map $\tilde{\varphi}$ of K_1 into K_2 defined by the formula

$$(10) \qquad \tilde{\varphi}(x) = \int t\, d\hat{\varphi}\mu(t), \qquad \mu \in \mathfrak{M}_x^+\,.$$

Clearly $\tilde{\varphi}$ is an affine extension of φ. To prove continuity, we shall first verify that $\tilde{\varphi}$ is weakly of the first Baire class, i.e. that $f \circ \tilde{\varphi}$ is of the first Baire class for every $f \in E_2^*$.

Assume f to be an arbitrary member of E_2^*. By a standard argument (based on the Hahn–Banach Theorem in $E_1 \times \mathsf{R}$ and on the existence of a countable base of open sets for the metrizable compact space K_1, cf. e.g. [11, ch. 3]), there exists a descending sequence $\{g_n\}_{n=1,2,\ldots}$ of continuous concave functions converging pointwise to $\overline{\varphi \circ f}$.

Let x be an arbitrary point in K_1. By the formula (3) and by the definition of barycenter, we shall have

$$(11) \qquad \int f \circ \varphi\, d\mu = \int f\, d\hat{\varphi}\mu = f(\tilde{\varphi}(x))\,,$$

for every $\mu \in \mathfrak{M}_x^+$.

By virtue of (9), we obtain for every natural number n

$$(12) \qquad \underline{g_n}(x) = \inf\left\{\int g_n\, d\mu \mid \mu \in \mathfrak{M}_x^+\right\}$$
$$\geqq \inf\left\{\int f \circ \varphi\, d\mu \mid \mu \in \mathfrak{M}_x^+\right\} = f(\tilde{\varphi}(x))\,.$$

On the other hand, we may apply (8) together with the Monotone Convergence Theorem and the identity of $\overline{f \circ \varphi}$ and $f \circ \varphi$ on $\partial_e K_1$ to yield the following series of relations in which μ denotes an arbitrary member of $\mathfrak{M}_x^+$:

$$(13) \quad \lim_{n\to\infty} \underline{g_n}(x) \leqq \lim_{n\to\infty} \int g_n\, d\mu = \int \overline{f \circ \varphi}\, d\mu = \int f \circ \varphi\, d\mu = f(\tilde{\varphi}(x))\,.$$

By combination of (12) and (13) we obtain

$$(14) \qquad \underline{g_n} \searrow f \circ \tilde{\varphi}\,.$$

By definition, $\underline{g_n}$ is lower semi-continuous for $n = 1, 2, \ldots$. Hence $\underline{g_n}$ belongs to the class $\mathscr{C}_\sigma(K_1)$ of all pointwise limits of ascending sequences from $\mathscr{C}(K_1)$. (Recall that K_1 has a countable base.) By (14), $f \circ \tilde{\varphi}$ belongs to the class $\mathscr{C}_{\sigma\delta}(K_1)$ of all pointwise limits of descending sequences from $\mathscr{C}_{\sigma}(K_1)$. By a dual argument, $f\circ\tilde{\varphi}$ also belongs to the dual class $\mathscr{C}_{\delta\sigma}(K_1)$. Clearly the first Baire class $\mathscr{C}_{\lambda}(K_1)$ (consisting of all pointwise limits of sequences from $\mathscr{C}(K_1)$) is contained in each of the two classes $\mathscr{C}_{\sigma\delta}(K_1)$ and $\mathscr{C}_{\delta\sigma}(K_1)$. By a classical lemma of Sierpinski [12, p. 13], it is equal to their intersection. Hence we shall have

$$f\circ\tilde{\varphi}\in\mathscr{C}_{\sigma\delta}(K_1)\cap\mathscr{C}_{\delta\sigma}(K_1)=\mathscr{C}_{\lambda}(K_1)\,, \tag{15}$$

for every $f\in E_2{}^*$. Thus $\tilde{\varphi}$ is weakly of the first Baire class.

The proof that $\tilde{\varphi}$ is continuous, is based on a general "lifting"-technique. Let $\mathscr{K}_1$ be the set of all positive normalized measures supported by $\overline{\partial_e K_1}$, let $\mathscr{K}_2$ be the set of all positive normalized measures on K_2, and define the "barycenter-map" ϱ_i from $\mathscr{K}_i$ to K_i as follows:

$$\varrho_i(\mu)=\int t\,d\mu(t),\qquad \mu\in\mathscr{K}_i,\ i=1,2\,. \tag{16}$$

Clearly $\mathscr{K}_i$ is a vaguely compact convex set (it is even an "r-simplex" in the terminology of [1]), and ϱ_i is a continuous map of $\mathscr{K}_i$ onto K_i for $i=1,2$.

Now let f be some member of $E_2{}^*$, let $x\in K_1$ and let μ be an arbitrary member of $\mathfrak{M}_x{}^+$. By virtue of (8)

$$\underline{f\circ\varphi}(x)\leqq\int f\circ\varphi\,d\mu\leqq\overline{f\circ\varphi}(x)\,,$$

and hence by (11)

$$\underline{f\circ\varphi}(x)\leqq f\circ\tilde{\varphi}(x)\leqq\overline{f\circ\varphi}(x)\,. \tag{17}$$

Let φ_0 be the restriction of $\tilde{\varphi}$ to $\overline{\partial_e K_1}$. By the Lemma, the equality sign holds good throughout (17) when $x\in\partial_e K_1$. By the requirement (ii), this implies

$$\underline{f\circ\varphi}(x)=f\circ\varphi_0(x)=\overline{f\circ\varphi}(x),\qquad x\in\overline{\partial_e K_1}; \tag{18}$$

and the function $f\circ\varphi_0$ on $\overline{\partial_e K_1}$ is continuous.

By compactness, the weak topology on K_2 coincides with the given topology on K_2. Hence φ_0 is a continuous map of $\overline{\partial_e K_1}$ into K_2. It follows (by use of (3)) that $\hat{\varphi}_0$ is a (vaguely) continuous map of $\mathscr{K}_1$ into $\mathscr{K}_2$.

We claim that the following diagram is commutative:

$$\begin{array}{ccc}\mathscr{K}_1 & \xrightarrow{\ \hat{\varphi}_0\ } & \mathscr{K}_2\\ \downarrow{\scriptstyle\varrho_1} & & \downarrow{\scriptstyle\varrho_2}\\ K_1 & \xrightarrow{\ \tilde{\varphi}\ } & K_2\end{array}$$

To prove this claim, we consider an element μ of $\mathscr{K}_1$ and an arbitrary $f \in E_2^*$. By a theorem of G. Choquet [7] (cf. [11, ch. 12] for detailed proof), the "barycenter formula" is valid for every affine function of the first Baire class. Applied to our function $f \circ \tilde{\varphi}$ on K_1 and to the given measure μ, this means

$$(f \circ \tilde{\varphi})(\varrho_1(\mu)) = \int f \circ \tilde{\varphi}\, d\mu \,. \tag{19}$$

Further, by definition, and by the barycenter formula for $f \in E_2^*$

$$(f \circ \tilde{\varphi})(\varrho_1(\mu)) = \int f \circ \varphi_0\, d\mu = \int f\, d\hat{\varphi}_0\mu = f(\varrho_2(\hat{\varphi}_0\mu)) \,. \tag{20}$$

Hence

$$f \circ \tilde{\varphi} \circ \varrho_1 = f \circ \varrho_2 \circ \hat{\varphi}_0 \,,$$

and since f was an arbitrary member of E_2^*, we shall have the desired commutativity

$$\tilde{\varphi} \circ \varrho_1 = \varrho_2 \circ \hat{\varphi}_0 \,. \tag{21}$$

Now let F be an arbitrary closed subset of K_2. Since ϱ_1 is surjective, we shall have

$$\tilde{\varphi}^{-1}(F) = \varrho_1[\hat{\varphi}_0^{-1}(\varrho_2^{-1}(F))] \,. \tag{22}$$

By the continuity of the maps occurring on the right hand side of (22), and by the compactness of $\mathscr{K}_1$, $\tilde{\varphi}^{-1}(F)$ is closed. Hence we have proved $\tilde{\varphi}$ to be a continuous affine extension of φ.

2) Assume $\tilde{\varphi}$ to be a continuous affine map of K_1 into K_2 which extends φ. Let $\nu \in \mathfrak{N}(\partial_e K_1)$ and assume (without lack of generality) that $\nu^+(K_1) = \nu^-(K_1) = 1$. Now ν^+ and ν^- are positive normalized boundary measures with common barycenter, say x. By the definition of barycenter, we shall have for every $f \in E_2^*$:

$$\int f\, d\hat{\varphi}\nu = \int f \circ \varphi\, d\nu = \int f \circ \varphi\, d\nu^+ - \int f \circ \varphi\, d\nu^- \tag{23}$$
$$= f(\tilde{\varphi}(x)) - f(\tilde{\varphi}(x)) = 0 \,.$$

Clearly also $\hat{\varphi}\nu(K_2) = 0$. Hence $\hat{\varphi}\nu \in \mathfrak{N}(K_2)$. The condition (ii) is also trivially satisfied, since $\underline{f \circ \varphi} = f \circ \tilde{\varphi} = \overline{f \circ \varphi}$ in the present case.

3) If $\partial_e K_1$ is closed, then the condition (ii) is satisfied by virtue of the Lemma. Moreover since $\partial_e K_1 = \overline{\partial_e K_1}$, the functions φ and φ_0 of the above proof coincide, and the crucial formula (20) obtained by an argument involving metrizability, now simply reduces to the definition of $\tilde{\varphi}$ (10).

COROLLARY. *Let K be a metrizable compact convex set in a locally convex space. A continuous and bounded real-valued function g on $\partial_e K$ can be* *extended to a continuous affine function on K if and only if the following two requirements are satisfied.*

(i)′ $$\int g\, d\nu = 0 \quad \text{for} \quad \text{all } \nu \in \mathfrak{N}(\partial_e K)$$

(ii)′ $$\bar{g} \text{ and } \underline{g} \text{ are continuous on } \overline{\partial_e K}\,.$$

If $\partial_e K$ is closed, then (ii)′ *is automatically satisfied, and the conclusion holds without metrizability.*

PROOF. Let K_2 be some compact interval containing $g(\partial_e K)$, and consider g as a map from $\partial_e K$ into K_2. Assume $\nu \in \mathfrak{N}(\partial_e K)$. Clearly $\hat{g}\nu$ has total mass zero, hence $\hat{g}\nu \in \mathfrak{N}(K_2)$ if and only if the resultant of $\hat{g}\nu$ is in the origin. By the definition of resultant and by the fact that the only (up to scalar multiples) continuous linear functional on R is $f(\xi) \equiv \xi$, we shall have $\bar{g}\nu \in \mathfrak{N}(K_2)$ if and only if

$$0 = \int f\, d\hat{g}\nu = \int f \circ g\, d\nu = \int g\, d\nu\,. \tag{24}$$

Hence the condition (i) of the Theorem is equivalent to the condition (i)′ of the Corollary.

Similarly, the equivalence of (ii) and (ii)′ follows by substitution of $\varphi = g$ and $f(\xi) \equiv \xi$ into the former of these two conditions.

The above Corollary gives a necessary and sufficient condition for solvability of the Dirichlet problem of the extreme boundary with respect to the class of continuous affine functions. By a standard technique it may be transferred to the Dirichlet problem for the Choquet boundary of a compact set X with respect to a uniformly closed linear subspace of $\mathscr{C}(X)$ containing the constants and separating points (cf. e.g. [11, ch. 6]). The last part of the Corollary generalizes a result of H. Bauer, by which every continuous function on the extreme boundary of a simplex with closed extreme boundary (an "r-simplex") is extendable to a continuous affine function on the whole simplex [3, p. 120]. The above Corollary is stated with a sketch of proof in the note [2].

The condition (ii) can not be omitted from the Theorem in the general case. In fact *there exist non-isomorphic simplexes with homeomorphic extreme boundaries.* In the proof of Theorem 1 of [1, p. 101] there is given an example of a simplex with extreme boundary homeomorphic to N (the set of integers with discrete topology). In [9, p. 29], E. Effros has given a similar example of a simplex with extreme boundary homeomorphic to N. However, the two simplexes are not isomorphic, as can be seen from the fact that their facial structures are entirely different (cf. [9] for details).

It is not hard to see that the condition (ii) implies uniform continuity of φ. (In the course of the proof this result was established by formula (18).) However, it is of some interest to note that uniform continuity alone cannot replace the property (ii). In this connection it suffices to consider the well-known Bourbaki example in Euclidean 3-space [5, p. 87, ex. 8]. If g is a real valued function with the value zero on the extreme points of the generating circle and the value 1 on the extreme points of the distinguished line-segment, then g is uniformly continuous, and it is annihilated by every $\nu \in \mathfrak{N}(\partial_e K)$, but it is clearly not extendable to any continuous affine function on the whole set. (Nor is $\bar{g}$ continuous at the only non-extreme point of $\overline{\partial_e K}$.)

The case of a non-metrizable initial set K_1 remains open. The extended function $\tilde{\varphi}$ is well defined (through (10)) whenever $\partial_e K$ is a Baire set (or even a "K-Souslin set" [8, p. 151]). However, our remaining proof invokes metrizability in an essential way to yield existence of the sequence $\{g_n\}$ occurring in (14). The only result known to us, which subsists in the completely general case, is the simple fact that every continuous map φ which is defined on $\overline{\partial_e K_1}$ and preserves (in the sense of (i)) the affine dependences on $\overline{\partial_e K_1}$, can be extended to a homomorphism of K_1 into K_2. This follows by a direct application of the "lifting technique" of the last part of the above proof.

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