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PARABOLIC DIFFERENCE OPERATORS

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Consider the initial-value problem $(u=u(x,t))$

$$\frac{\partial u}{\partial t} = P(D)u, \qquad t>0 , \tag{1}$$

$$u(x,0) = u_0(x) , \tag{2}$$

where $P(D)$ is a partial differential operator with respect to x with constant matrix coefficients. There are a number of different definitions of parabolic systems in the literature; for the purpose of this paper we will say that the system (1) is parabolic if the initial-value problem (1), (2) is correctly posed in L^2 in the sense of Lax and Richtmyer (cf. [13]) that is if for any $u_0 \in L^2$ there exists exactly one (generalized) solution $u(x,t)=E(t)u_0(x)$ of (1), (2) and where

$$\sup\{\|E(t)\|;\ 0 \leqq t \leqq T\} < \infty$$

for any $T>0$, and if in addition for any partial differential operator $Q(D)$ with constant coefficients and any τ, T, with $0<\tau<T$ we have

$$\sup\{\|Q(D)E(t)\|;\ \tau \leqq t \leqq T\} < \infty .$$

This last condition means loosely speaking that the solution $E(t)u_0$ is very smooth for $t>0$ even if u_0 is not. It will be proved that if (1), (2) is correctly posed, then (1) is parabolic if and only if there are positive constants C_1, C_2, μ such that

$$\operatorname{Re}\lambda(\xi) \leqq -C_1|\xi|^\mu + C_2 , \tag{3}$$

where $\lambda(\xi)$ is any eigenvalue of $P(\xi)$ (Theorem 1.2). It will also be proved that (1) is parabolic if and only if there is a family of positive hermitean matrices $H(\xi)$ which are bounded away from zero and infinity such that for some positive constants C_1, C_2, μ,

$$\operatorname{Re}(H(\xi)P(\xi)) \leqq (-C_1|\xi|^\mu + C_2)I \tag{4}$$

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(Theorem 1.3). This characterization contains at the same time the correctness of (1), (2) and the condition (3), and is related to Kreiss' characterization of correctly posed problems [11]. If μ in (3) can be taken as the order p of $P(D)$, the system (1) is said to be parabolic in Petrowsky's sense; the correctness of (1), (2) is then automatic. A trivial example of a parabolic equation which is not parabolic in Petrowsky's sense is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x^3},$$

where $p=3$ and

$$\operatorname{Re} P(\xi) = \operatorname{Re}[(i\xi)^2 + (i\xi)^3] = -\xi^2 .$$

Consider consistent finite difference approximations $v_h(x,nk)$ to the solution $u(x,t)$ of (1), (2) defined recursively by

$$(5) \quad v_h(x,(n+1)k) = E_h v_h(x,nk) = A_h^{-1} B_h v_h(x,nk), \qquad n=0,1,\ldots ,$$

$$(6) \qquad v_h(x,0) = u_0(x) ,$$

where A_h and B_h are explicit difference operators which are polynomials in h and where k/h^p is constant. We will say that E_h is parabolic if it is stable, that is if for any $T>0$,

$$\sup\{\|E_h^n\|;\ 0 \leqq nk \leqq T\} < \infty ,$$

and if in addition for any difference operator Q_h consistent with a differential operator $Q(D)$ with constant coefficients and any τ, T with $0<\tau<T$ we have

$$\sup\{\|Q_h E_h^n\|;\ \tau \leqq nk \leqq T\} < \infty .$$

It will be proved that for parabolic systems (1), parabolicity of E_h is the necessary and sufficient condition in order that the family of functions v_h defined in (5), (6) has the property that for any Q_h consistent with a $Q(D)$, $Q_h v_h(x,nk)$ converges in L^2 to $Q(D)E(t)u_0$ as $k \to 0$ and $nk \to t$ (Theorem 2.2). It can further be proved that the rate of convergence is h^m where m is the minimum of the orders of accuracy of Q_h and E_h if u_0 has $m+p$ derivatives in L^2 (Theorem 2.3).

Introducing the amplification matrix or symbol $E_h(\xi)$ of the operator E_h we give a characterization of parabolic E_h analogous to (3): if E_h is stable, then it is parabolic if and only if there are positive constants C_1, C_2, ν such that

$$(7) \qquad \varrho_h(\xi) \leqq 1 - C_1 k |\xi|^\nu + C_2 k, \qquad h|\xi_j| \leqq \pi ,$$

where $\varrho_h(\xi)$ is the spectral radius of $E_h(\xi)$ (Theorem 3.2). It will also be proved that E_h is parabolic if and only if there is a family of positive hermitean matrices $H_h(\xi)$ which are bounded away from zero and infinity such that for some positive constants C_1, C_2, ν,

$$|E_h(\xi)|_{H_h(\xi)} \leqq 1 - C_1 k|\xi|^\nu + C_2 k, \qquad h|\xi_j| \leqq \pi , \tag{8}$$

where $|E|_H$ is the matrix-norm defined by the vector-norm $|x|_H = (Hx,x)^{\frac{1}{2}}$ (Theorem 3.4). This characterization contains at the same time the stability of E_h and condition (7), and is related to Kreiss' stability theorem [10]. The criteria (4) and (8) makes it possible to construct parabolic difference operators E_h consistent with any parabolic system (1) (Theorem 3.5).

The assumption of constant coefficients makes it possible to work throughout with Fourier transforms; the most important tool in proving (7) is the Seidenberg–Tarski elimination theorem.

Most of the previous work on difference methods for parabolic equations has been on second order equations (cf. e.g. [7], [1], [4], [14], and references). A condition of the type (7) appeared for the first time in connection with parabolic difference equations in the famous paper by John [7]. Recently, Aronsson [2] and Widlund [15], [16] have studied the general case of higher order systems with variable coefficients which are parabolic in Petrowsky's sense. Widlund thereby takes (7) with $\nu = p$ as a definition of a parabolic difference operator (he actually considers multistep schemes) and proves the stability of such an operator (also in the maximum-norm in [16]). An interesting application of a condition of the form (7) to the stability of difference approximations to hyperbolic equations has been given by Kreiss [12].

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1. Parabolic differential equations.

Consider the initial-value problem

$$\frac{\partial u}{\partial t} = P(D)u \equiv \sum_{|\alpha| \leq p} P_\alpha D^\alpha u, \qquad t > 0 , \tag{1.1}$$

$$u(x,0) = u_0(x) , \tag{1.2}$$

where $x = (x_1, \ldots, x_d) \in R^d$, $u = u(x,t)$ is a complex N-vector, and P_α are constant $N \times N$ matrices, $\alpha = (\alpha_1, \ldots, \alpha_d)$, $|\alpha| = \sum_j \alpha_j$, and

$$D^\alpha = i^{-|\alpha|}\, \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d} .$$

For complex N-vectors u, v we use the scalar product $(u,v) = \sum_j u_j \bar{v}_j$ and the euclidean norm $|u| = (\sum_j |u_j|^2)^{\frac{1}{2}}$, and for $N \times N$ matrices the corresponding norm

$$|A| = \sup\{|Au|;\ |u| = 1\}.$$

We denote by L^2 the set of complex N-vectors $u(x)$, $x \in R^d$, which are square integrable, and let

$$\|u\| = \left(\int |u(x)|^2\,dx\right)^{\frac{1}{2}}.$$

Correspondingly, if E is a bounded linear operator on L^2, we set

$$\|E\| = \sup\{\|Eu\|;\ \|u\| = 1\}.$$

We denote by $\mathscr{S}$ the set of infinitely differentiable N-vectors such that for any j and α,

$$\sup\{|x|^j\,|D^\alpha u(x)|;\ x \in R^d\} < \infty.$$

We have $\overline{\mathscr{S}} = L^2$.

We say that the initial-value problem (1.1), (1.2) is correctly posed (in L^2, in the sense of Lax and Richtmyer, cf. [13]) if there is a family of bounded linear operators $E_0(t)$, $t \geqq 0$, defined on a set $\mathscr{D} \subseteq \mathscr{S}$ with $\overline{\mathscr{D}} = L^2$ such that for $u_0 \in \mathscr{D}$ the problem (1.1), (1.2) has the unique solution $u(x,t) = E_0(t)u_0 \in \mathscr{S}$ (for fixed t) and

$$\sup\{\|E_0(t)\|;\ 0 \leqq t \leqq T\} < \infty \tag{1.3}$$

for all $T \geqq 0$. The operator $E_0(t)$ is called a solution operator.

If the initial-value problem (1.1), (1.2) is correctly posed we can of course extend the definition of $E_0(t)$ to a bounded operator $E(t)$ defined in the whole of L^2 by closure. The operator $E(t)$ is called the generalized solution operator. When we want to emphasize the dependence of $E(t)$ upon $P(D)$ we write $E(t) = E(t; P)$.

The operators $E(t)$, $t \geqq 0$, clearly enjoy the semi-group property

$$\begin{aligned} E(s+t) &= E(s)\,E(t), \qquad s,t \geqq 0, \\ E(0) &= I. \end{aligned} \tag{1.4}$$

If (1.1), (1.2) is correctly posed it follows from (1.3) and (1.4) that there are positive constants C_1, C_2 such that

$$\|E(t)\| \leqq C_1 \exp(tC_2), \qquad t \geqq 0.$$

In the sequel C will denote a positive constant, not necessarily the same at different occurances. When desirable for clarity, we will index the constants locally and write C_1, C_2, etc.

Defining the Fourier transform of a vector $u(x) \in \mathcal{S}$ by

$$\hat{u}(\xi) = (2\pi)^{-\frac{1}{2}d} \int u(x) \exp(-i\langle x,\xi\rangle)\, dx, \qquad \langle x,\xi\rangle = \sum_{j=1}^{d} x_j \xi_j ,$$

the problem (1.1), (1.2) reduces for $u(x,t) \in \mathcal{S}$ to the following initial-value problem for a system of ordinary differential equations with ξ as a parameter, namely

$$\frac{d\hat{u}(\xi,t)}{dt} = P(\xi)\, \hat{u}(\xi,t) ,$$

$$\hat{u}(\xi,0) = \hat{u}_0(\xi) ,$$

where $P(\xi) = \sum_{|\alpha| \leq p} P_\alpha \xi^\alpha$, $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_d^{\alpha_d}$. This problem has the solution

(1.5) $$\hat{u}(\xi,t) = \exp(tP(\xi))\, \hat{u}_0(\xi) ,$$

and we have:

LEMMA 1.1. *The initial-value problem* (1.1), (1.2) *is correctly posed if and only if for any* $T > 0$,

(1.6) $$\sup\{|\exp(tP(\xi))|;\ 0 \leqq t \leqq T,\ \xi \text{ real}\} < \infty .$$

PROOF. The Parseval relation, $\|\hat{u}\| = \|u\|$, with (1.5), proves that for any family of operators $E_0(t)$, $t \geqq 0$, satisfying the above definition of a correctly posed problem, we must have

$$\|E_0(t)\| = \sup\{|\exp(tP(\xi))|;\ \xi \text{ real}\} ,$$

which proves the necessity of (1.6). On the other hand, if (1.6) holds, let $\mathcal{D}$ be the set $\hat{C}_0^\infty$ of Fourier transforms of infinitely differentiable vector-functions with compact support. Then $\overline{\mathcal{D}} = L^2$ and (1.5) defines for each $t \geqq 0$ and $u_0 \in \mathcal{D}$ a function $u(x,t) = E_0(t) u_0$ with the required properties.

As above, condition (1.6) can be written

(1.7) $$|\exp(tP(\xi))| \leqq C_1 \exp(tC_2), \qquad t \geqq 0 ,$$

for some positive constants C_1, C_2.

For a $N \times N$ matrix A with eigenvalues λ_j, $j = 1,\ldots,N$, we define

$$\Lambda(A) = \max_j \operatorname{Re} \lambda_j .$$

A necessary condition for (1.6) to hold is then that (1.1), (1.2) is correctly posed in Petrowsky's sense, namely

(1.8) $$\sup\{\Lambda(P(\xi));\ \xi \text{ real}\} < \infty .$$

This follows at once from the fact that $\exp\big(t\Lambda(P(\xi))\big)$ is the spectral radius of $\exp(tP(\xi))$ so that

$$\exp\big(t\Lambda(P(\xi))\big) \leqq |\exp(tP(\xi))| \leqq C_1 \exp(C_2 t), \qquad t \geqq 0 .$$

It is also known that (1.8) is not sufficient for (1.1), (1.2) to be correctly posed. Necessary and sufficient conditions for (1.6) or (1.7) to hold have been given by Kreiss [11]; we will give below one such set of conditions.

For a $N \times N$ matrix A we define

$$\operatorname{Re} A = \tfrac{1}{2}(A + A^*) .$$

Clearly $\operatorname{Re} A$ is a hermitean matrix. For hermitean $N \times N$ matrices, $A \leqq B$ means $(Au,u) \leqq (Bu,u)$ for all N-vectors u. The identity matrix is denoted by I.

We now state a modification of a well-known lemma by Kreiss [9].

LEMMA 1.2. *Let $\mathscr{F}$ denote a family of $N \times N$ matrices. Then*

$$\sup\{|\exp(At)|;\ A \in \mathscr{F}, t \geqq 0\} < \infty \tag{1.9}$$

if there is a positive constant C and for any $A \in \mathscr{F}$ a hermitean matrix H with

$$C^{-1} I \leqq H \leqq C I \tag{1.10}$$

and

$$\operatorname{Re}(HA) \leqq 0 .$$

On the other hand, if (1.9) *holds and $0 \leqq \gamma < 1$ there is a constant C, and for any $A \in \mathscr{F}$ a hermitean matrix H satisfying* (1.10) *and*

$$\operatorname{Re}(HA) \leqq \gamma \Lambda(A) H \leqq 0 .$$

PROOF. With $\gamma = 0$ in the second part, the lemma is proved in [9]. Our modification is easily proved by reviewing the proof in [9]. A similar modification of the discrete analogue of Lemma 1.2 was used by Widlund [16] (cf. Lemma 3.1 below).

From this lemma we easily obtain:

THEOREM 1.1. *The problem* (1.1), (1.2) *is correctly posed if there are positive constants C_1, C_2 and for each real ξ a hermitean matrix $H(\xi)$ such that*

$$C_1^{-1} I \leqq H(\xi) \leqq C_1 I \tag{1.11}$$

and

$$\operatorname{Re}(H(\xi)P(\xi)) \leqq C_2 I .$$

On the other hand, if (1.1), (1.2) *is correctly posed we have* (1.8) *and there* *are positive constants* C_1, C_2, C_3 *and for each real* ξ *a positive definite hermitean matrix* $H(\xi)$ *satisfying* (1.11) *and*

$$\text{Re}(H(\xi)P(\xi)) \leqq [C_2 + C_3\Lambda(P(\xi))]I. \tag{1.12}$$

There are many different definitions of a parabolic system of the form (1.1) in the literature. As our definition we will take the regularity with respect to x of the solution $E(t)u_0$ of the initial-value problem (1.1), (1.2) for $t>0$. More precisely, we say that the system (1.1), or the generalized solution operator $E(t)=E(t;P)$, is parabolic (in L^2) if (1.1), (1.2) is correctly posed and if for any (scalar) partial differential operator $Q(D)$ with respect to x and any τ, T with $0<\tau<T$ we have for a corresponding solution operator $E_0(t)$

$$\sup\{\|Q(D)E_0(t)\|,\ \tau \leqq t \leqq T\} < \infty. \tag{1.13}$$

Clearly, we could have replaced the solution operator $E_0(t)$ by the generalized solution operator $E(t)$ in (1.13); the generalized solution is indeed in this case infinitely differentiable for $t>0$. Further, it is easily seen that we would have arrived at an equivalent definition if we had assumed (1.13) only for first order operators $Q(D)$. We have

LEMMA 1.3. *The operator* $E(t;P)$ *is parabolic if and only if* (1.6) *holds for any* $T>0$ *and if for any (scalar) polynomial* $Q(\xi)$ *of degree* $q>0$ *and any* τ, T *with* $0<\tau<T$

$$\sup\{|Q(\xi)\exp(tP(\xi))|;\ \tau \leqq t \leqq T,\ \xi \text{ real}\} < \infty. \tag{1.14}$$

PROOF. Follows as in the proof of Lemma 1.1 by Fourier transforms.

In the sequel we will often for convenience write conditions (1.6) and (1.14) together in the form: if $Q(\xi)$ is any polynomial of degree $q \geqq 0$ and $\tau>0$, then

$$\sup\{|Q(\xi)\exp(tP(\xi))|;\ q\tau \leqq t \leqq T\} < \infty.$$

Our next aim is to give algebraic characterizations of parabolic operators. We have

THEOREM 1.2. *Assume that the initial-value problem* (1.1), (1.2) *is correctly posed. Then* $E(t;P)$ *is parabolic if and only if there are positive constants* C_1, C_2, μ *such that for all real* ξ,

$$\Lambda(P(\xi)) \leqq -C_1|\xi|^{\mu} + C_2. \tag{1.15}$$

For the proof we need some lemmas:

LEMMA 1.4. *If* A *is a* $N \times N$ *matrix, we have for* $t \geqq 0$,

$$|\exp(tA)| \leqq \exp(t\Lambda(A)) \sum_{j=0}^{N-1} (2t|A|)^j .$$

PROOF. See Gelfand–Šilov [6, p. 64].

LEMMA 1.5. *Let $\tilde{\Lambda}(r) = \max_{|\xi|=r} \Lambda(P(\xi))$. Then $\tilde{\Lambda}(r)$ is an algebraic function of r for large r.*

PROOF. See Friedmann [5, p. 219]. The essential tool in the proof is the Seidenberg–Tarski elimination theorem, which will be stated and used in Section 3 below (Lemma 3.3).

We can now give the

PROOF OF THEOREM 1.2. We first prove the sufficiency of the condition (1.15) for parabolicity. Thus assume that (1.15) holds. We then have by Lemma 1.4, for $0 < \tau \leqq t \leqq T$,

$$\begin{aligned} |\xi^\alpha \exp(tP(\xi))| &\leqq |\xi^\alpha| \exp\big(t\Lambda(P(\xi))\big) \sum_{j=0}^{N-1} (2t|P(\xi)|)^j \\ &\leqq C(1+T)^{N-1}(1+|\xi|)^{|\alpha|+(N-1)p} \exp(-\tau C_1|\xi|^\mu) , \end{aligned}$$

which is bounded and so, by Lemma 1.3, $E(t;P)$ is parabolic.

We shall now prove the necessity part of Theorem 1.2 and assume that $E(t;P)$ is parabolic. Then by Lemma 1.3,

$$(1+|\xi|) \exp\big(t\Lambda(P(\xi))\big) \leqq (1+|\xi|)\, |\exp(tP(\xi))| \leqq C ,$$

for $0 < \tau \leqq t \leqq T$ and so, by taking maximum over $|\xi| = r$ and setting $t = 1$,

$$(1+r) \exp(\tilde{\Lambda}(r)) \leqq C ,$$

or

$$\tilde{\Lambda}(r) \leqq \log C - \log(1+r) . \tag{1.16}$$

By Lemma 1.5, $\tilde{\Lambda}(r)$ is algebraic in r for large r and so by developing $\tilde{\Lambda}(r)$ in a Puiseux series around $r = \infty$ there is a (rational) number μ and a constant C_1 such that

$$\tilde{\Lambda}(r) = -2C_1 r^\mu(1+o(1)), \quad r \to \infty .$$

Because of (1.16) we must have $C_1 > 0$, $\mu > 0$ and so for sufficiently large r,

$$\tilde{\Lambda}(r) \leqq -C_1 r^\mu .$$

This clearly proves (1.15).

It follows from the proof of Theorem 1.2 that the condition (1.15) is equivalent to the condition (1.14) with $\tau > 0$. In the case where the matrix $P(\xi)$ is normal, in particular in the scalar case ($N = 1$), the condition (1.15) is sufficient for parabolicity; it is then not necessary to explicitly assume that the initial-value problem is correctly posed since in that case for $0 \leqq t \leqq T$, with the C_2 in (1.15),

$$|\exp(tP(\xi))| = \exp\bigl(t\Lambda(P(\xi))\bigr) \leqq \exp(TC_2) .$$

Also, the condition (1.15) with $\mu = p$ is sufficient for parabolicity since in that case by Lemma 1.4 we have for $0 \leqq t \leqq T$,

$$\begin{aligned} |\exp(tP(\xi))| &\leqq \exp\bigl(t\Lambda(P(\xi))\bigr) \sum_{j=0}^{N-1} (2t|P(\xi)|)^j \\ &\leqq C\bigl(1 + (t|\xi|^p)^{N-1}\bigr) \exp(-C_1 t|\xi|^p) \leqq C . \end{aligned}$$

In this case, the operator $E(t;P)$ is said to be parabolic in Petrowsky's sense. In the general case, however, the correctness of the intial-value problem has to be explicitly assumed. To see this we consider the example where $P(\xi)$ is the matrix

$$P(\xi) = \begin{pmatrix} -\xi^2 & \xi^4 \\ 0 & -\xi^2 \end{pmatrix} = -\xi^2 I + \xi^4 J .$$

We obtain $\Lambda(P(\xi)) = -\xi^2$ so that condition (1.15) is satisfied with $\mu = 2$. However, we obtain

$$\exp(tP(\xi)) = \exp(-t\xi^2)\,(I + \xi^4 tJ)$$

and this matrix is not bounded for $0 \leqq t \leqq T$ for any $T > 0$ (set $t = \xi^{-2}$ and let $|\xi| \to \infty$), and thus $E(t;P)$ is not parabolic.

It follows from the proof of Theorem 1.2 that for a parabolic operator $E(t;P)$ there is a largest μ for which (1.15) holds. This number we call the order of parabolicity of $E(t;P)$. In particular, an operator which is parabolic in Petrowsky's sense is an operator which is parabolic of order p.

Operators of the form $E(t;P)$ which satisfy (1.15) for some positive constants C_1, C_2, μ are called parabolic in Šilov's sense, even if the initial-value problem is not correctly posed in the above sense (cf. [5] or [6]). Our concept of parabolicity is thus in general more restrictive than Šilov's.

We conclude this section by a characterization of parabolic operators which contains at the same time the correctness of the initial-value problem and the condition (1.15).

THEOREM 1.3. *The operator $E(t;P)$ is parabolic of order at least μ if and only if there are positive constants C_1, C_2, C_3 and for each real ξ a hermitean matrix $H(\xi)$ such that*

$$C_1^{-1} I \leqq H(\xi) \leqq C_1 I \tag{1.17}$$

and

$$\operatorname{Re}\bigl(H(\xi)P(\xi)\bigr) \leqq (-C_2|\xi|^\mu + C_3) I . \tag{1.18}$$

PROOF. It follows from Theorem 1.1 that (1.17) and (1.18) imply that the initial-value problem is correctly posed. Since for any positive definite $N \times N$ matrix H and any $N \times N$ matrix P,

$$\Lambda(P) \leqq \sup_{u \neq 0} \frac{(\operatorname{Re}(HP)u,u)}{(Hu,u)},$$

condition (1.18) implies (1.15) and so $E(t;P)$ is parabolic of order at least μ.

On the other hand, if $E(t;P)$ is parabolic of order at least μ, (1.15) holds and by Theorem 1.1 there is a hermitean matrix $H(\xi)$ satisfying (1.17) and (1.12). Together, (1.12) and (1.15) give (1.18).

2. Parabolic difference operators.

For the approximate solution of the initial-value problem (1.1), (1.2) we consider operators of the form

$$(A_h v)(x) = \textstyle\sum_\beta a_\beta(h)\, v(x+\beta h)\,, \tag{2.1}$$

where h is a small positive parameter, $\beta = (\beta_1,\ldots,\beta_d)$ with β_j integer, $a_\beta(h)$ are $N \times N$ matrices which are polynomials in h, and the summation is over a finite set of β. Such an operator is referred to as an explicit difference operator. We introduce the symbol of the operator A_h,

$$A_h(\xi) = \textstyle\sum_\beta a_\beta(h) \exp(i\langle\beta,h\xi\rangle)\,,$$

which is periodic with period $2\pi/h$ in ξ_j, and notice that for $v \in L^2$, the Fourier transform of $A_h v$ is

$$\widehat{(A_h v)}(\xi) = A_h(\xi)\, \hat{v}(\xi)\,.$$

Further A_h^{-1} exists and is a bounded operator in L^2 if and only if $\det A_h(\xi) \neq 0$ for all real ξ. We then have

$$\|A_h^{-1}\| = \sup\{|A_h(\xi)^{-1}|;\ \xi \text{ real}\}\,.$$

In particular, if

$$\det A_h(h^{-1}\xi)\,\big|_{h=0} = \det\left(\textstyle\sum_\beta a_\beta(0) \exp(i\langle\beta,\xi\rangle)\right) \neq 0, \qquad \xi \text{ real}\,, \tag{2.2}$$

then $\det A_h(\xi)$ is bounded away from zero for sufficiently small h, say $h \leqq h_0$, and all real ξ, and it follows that A_h^{-1} is uniformly bounded for $h \leqq h_0$.

We then replace the equation (1.1) by a finite difference equation

$$A_h v(x,t+k) = B_h v(x,t)\,, \tag{2.3}$$

where A_h and B_h are explicit operators of the form (2.1) such that (2.2) holds for A_h that is A_h^{-1} exists and is uniformly bounded for $h \leqq h_0$, and where k is a positive parameter tied to h by the relation $k/h^p = \lambda =$ constant. The relation (2.3) can be solved for $v(x,t+k)$ if $h \leqq h_0$,

$$(2.4) \qquad v(x,t+k) = A_h^{-1}B_h v(x,t) = E_h v(x,t) ,$$

and we want to use (2.4) for step-wise computation of an approximate solution $v(x,nk)$, $n=1,2,\ldots$, of the initial-value problem (1.1), (1.2) by setting

$$v(x,nk) = E_h^n u_0(x) .$$

The operator E_h defined by (2.4) is then supposed to be chosen so as to approximate the solution operator $E(k)=E(k;P)$ introduced in Section 1 if this operator exists: we say that E_h is consistent with the equation (1.1), or, by abuse of the language, with $E(t;P)$, if for any infinitely differentiable solution $u(x,t)$ of (1.1),

$$(2.5) \qquad A_h u(x,t+k) = B_h u(x,t)+o(k), \quad k \to 0 .$$

Notice that this definition does not assume that the initial-value problem is correctly posed, or that $E(t;P)$ exists, and that it only uses the solution $u(x,t)$ in a small neighborhood of the point (x,t).

We introduce the symbol of the operator E_h,

$$(2.6) \qquad E_h(\xi) = A_h(\xi)^{-1} B_h(\xi) = \textstyle\sum_\beta e_\beta(h) \exp(i\langle\beta,h\xi\rangle) ,$$

and obtain for the Fourier transform of $E_h^n v$ if $v \in L^2$,

$$(2.7) \qquad (\widehat{E_h^n v})(\xi) = E_h(\xi)^n \hat{v}(\xi) .$$

Clearly, since $A_h(\xi)$ and $B_h(\xi)$ are polynomials in h, $\exp(ih\xi_j)$, $j=1,\ldots,d$, we have that $A_h(\xi)^{-1}$ and $E_h(\xi)$ are analytic in h, $h \leqq h_0$, and ξ. In particular, the series (2.6) is absolutely convergent and the operator E_h can also be defined by

$$E_h v(x) = \textstyle\sum_\beta e_\beta(h)\, v(x+\beta h) .$$

If this sum is infinite, that is if taking the inverse of A_h is not a trivial operation, the operator E_h is said to be an implicit difference operator.

It is well known that the consistency condition (2.5) can be expressed in terms of the symbol $E_h(\xi)$ of the operator E_h:

LEMMA 2.1. *The operator E_h is consistent with $E(t;P)$ if and only if*

$$E_h(h^{-1}\xi) = \exp(kP(h^{-1}\xi)+o(k+|\xi|^p)), \quad k,\xi \to 0 .$$

Proof. Let $a_\beta(h)$ and $b_\beta(h)$ be the polynomials

$$a_\beta(h) = \sum_j a_{\beta j} h^j, \qquad b_\beta(h) = \sum_j b_{\beta j} h^j .$$

Developing both sides of (2.5) in Taylor series around the point (x,t) we get by (1.1),

$$\sum_{j+|\alpha|\leqq p} h^{j+|\alpha|} (\sum_\beta \beta^\alpha a_{\beta j}) \frac{i^{|\alpha|}}{\alpha!} D^\alpha u(x,t) + \lambda h^p (\sum_\beta a_{\beta 0}) P(D) u(x,t)$$
$$= \sum_{j+|\alpha|\leqq p} h^{j+|\alpha|} (\sum_\beta \beta^\alpha b_{\beta j}) \frac{i^{|\alpha|}}{\alpha!} D^\alpha u(x,t) + o(h^p), \quad h \to 0 .$$

Since the $D^\alpha u(x,t)$ are arbitrary we conclude that (2.5) holds if and only if

$$\sum_\beta \beta^\alpha a_{\beta j} = \sum_\beta \beta^\alpha b_{\beta j}, \quad j+|\alpha|<p ,$$
$$\sum_\beta \beta^\alpha a_{\beta j} + \lambda i^{-|\alpha|} \sum_\beta a_{\beta 0} P_\alpha = \sum_\beta \beta^\alpha b_{\beta j}, \quad j+|\alpha|=p .$$

On the other hand these are the necessary and sufficient conditions that the Maclaurin expansions of the analytic functions

$$A_h(h^{-1}\xi) \exp(kP(h^{-1}\xi)) \quad \text{and} \quad B_h(h^{-1}\xi)$$

as functions of (h,ξ) have the same coefficients for $h^j \xi^\alpha$ when $j+|\alpha| \leqq p$, that is

$$A_h(h^{-1}\xi) \exp(kP(h^{-1}\xi)) = B_h(h^{-1}\xi) + o(h^p + |\xi|^p), \quad h,\xi \to 0 .$$

Since $A_h(\xi)^{-1}$ is uniformly bounded for $h \leqq h_0$, ξ real, this proves the lemma.

Assume that the initial-value problem (1.1), (1.2) is correctly posed so that $E(t;P)$ exists and is uniformly bounded in any finite interval $[0,T]$. We then say that the operator E_h converges to $E(t)=E(t;P)$ (when $h \to 0$) if for any $u_0 \in L^2$, $t \geqq 0$, and any pair of sequences $\{h_j\}_{j=1}^\infty$, $\{n_j\}_{j=1}^\infty$ with $h_j \to 0$, $n_j k_j \to t$ as $j \to \infty$,

$$\|E_{h_j}^{n_j} u_0 - E(t)u_0\| \to 0, \quad j \to \infty . \tag{2.8}$$

It is well known that consistency alone is not sufficient to guarantee convergence: we say that E_h is stable if for any $T>0$,

$$\sup\{\|E_h^n\|;\ 0 \leqq nk \leqq T\} < \infty , \tag{2.9}$$

and we have the Lax equivalence theorem (cf. [13] where it is proved in a more general framework):

Theorem 2.1. *Assume that E_h is consistent with $E(t)=E(t;P)$ and that (1.1), (1.2) is correctly posed. Then E_h converges to $E(t)$ if and only if E_h is stable.*

PROOF. We first prove the necessity of stability for convergence. Thus assume that E_h converges to $E(t)$ and that (2.9) does not hold. Then there are sequences $\{h_j\}_{j=1}^{\infty}$, $\{n_j\}_{j=1}^{\infty}$ and a t with $0 \leqq t \leqq T$ such that $n_j k_j \to t$ when $j \to \infty$ and

$$\|E_{h_j}^{n_j}\| \to \infty, \quad j \to \infty . \tag{2.10}$$

On the other hand, for any $u_0 \in L^2$ we have by the convergence of E_h to $E(t)$,

$$\|E_{h_j}^{n_j} u_0\| \to \|E(t)u_0\|, \quad j \to \infty ,$$

so that $\{\|E_{h_j}^{n_j} u_0\|\}_{j=1}^{\infty}$ is a bounded sequence for any $u_0 \in L^2$. But then, by Banach–Steinhaus' theorem $\{\|E_{h_j}^{n_j}\|\}_{j=1}^{\infty}$ is a bounded sequence. This contradicts (2.10) and so E_h is stable.

We now prove the sufficiency of stability for convergence. Because of the uniform boundedness of $E_{h_j}^{n_j}$ and the boundedness of $E(t)$ it is sufficient to prove (2.8) for the dense subset $\hat{C}_0^{\infty}$ of L^2. But by (1.5), (2.7), and Parseval's relation,

$$\begin{aligned}\|E_{h_j}^{n_j} u_0 - E(t)u_0\| &= \|[E_{h_j}(\xi)^{n_j} - \exp(tP(\xi))]\, \hat{u}_0(\xi)\| \\ &\leqq \|u_0\| \sup\{|E_{h_j}(\xi)^{n_j} - \exp(tP(\xi))|;\ \hat{u}_0(\xi) \neq 0\} ,\end{aligned}$$

and the result follows at once from the following lemma:

LEMMA 2.2. *If E_h is consistent with $E(t;P)$, then*

$$E_{h_j}(\xi)^{n_j} \to \exp(tP(\xi)), \quad h_j \to 0,\ n_j k_j \to t , \tag{2.11}$$

uniformly on compact sets in ξ.

PROOF. We have by Lemma 2.1,

$$E_h(\xi) = \exp(kP(\xi) + o(k)), \quad k \to 0 ,$$

uniformly on compact sets in ξ. This implies, again uniformly on compact sets in ξ,

$$\begin{aligned}E_{h_j}(\xi)^{n_j} &= \exp(n_j k_j P(\xi) + o(1)) \\ &= \exp(tP(\xi) + o(1)), \quad h_j \to 0,\ n_j k_j \to t ,\end{aligned}$$

which is (2.11).

The stability of E_h can be expressed in terms of the symbol:

LEMMA 2.3. *The operator E_h is stable if and only if for any $T > 0$,*

$$\sup\{|E_h(\xi)^n|;\ 0 \leqq nk \leqq T,\ \xi \text{ real}\} < \infty .$$

PROOF. Follows at once from (2.9) and

$$\|E_h^n\| = \sup\{|E_h(\xi)^n|;\ \xi \text{ real}\} .$$

An immediate consequence of Lemmas 2.2 and 2.3 is:

LEMMA 2.4. *A necessary condition for the stability of the operator* E_h, *consistent with* $E(t;P)$ *is that* (1.1), (1.2) *is correctly posed.*

Beside operators of the form (2.1) approximating the solution operator $E(t)$ we shall also consider operators of similar form approximating differential operators with constant coefficients in x: we say that the operator

$$Q_h v(x) = h^{-q} \sum_\beta q_\beta(h)\, v(x+\beta h) , \tag{2.12}$$

where the sum is finite and the scalar functions $q_\beta(h)$ are polynomials in h, is consistent with the differential operator

$$Q(D)v = \sum_{|\alpha| \leqq q} Q_\alpha D^\alpha v$$

of order q, if for any infinitely differentiable function v,

$$Q_h v(x) = Q(D)\, v(x) + o(1), \quad h \to 0 .$$

We introduce in the same manner as before the symbol

$$Q_h(\xi) = h^{-q} \sum_\beta q_\beta(h) \exp(i\langle\beta, h\xi\rangle) ,$$

and have:

LEMMA 2.5. *The operator* Q_h *is consistent with* $Q(D)$ *if and only if*

$$h^q Q_h(h^{-1}\xi) = h^q Q(h^{-1}\xi) + o(h^q + |\xi|^q), \quad h, \xi \to 0 . \tag{2.13}$$

PROOF. Analogous to the proof of Lemma 2.1.

The two concepts of consistency are clearly related by the fact that E_h is consistent with $E(t;P)$ if and only if $k^{-1}(E_h - I)$ is consistent with $P(D)$.

We now introduce the following definition: we say that the operator E_h is parabolic if for any operator Q_h of the form (2.12), consistent with a differential operator $Q(D)$ of order q and for any τ, T with $\tau > 0$ we have

$$\sup\{\|Q_h E_h^n\|;\ q\tau \leqq nk \leqq T\} < \infty . \tag{2.14}$$

Notice that as in the continuous case by setting $Q_h = I$, (2.14) contains stability. In Section 3 we will give an example which shows that replacing $q\tau$ by τ in (2.14) would not have given an equivalent definition. It will be convenient in the sequel to assume that if $q = 0$, then Q_h is always constant, independent of h.

The introduction of parabolic difference operators is motivated by the following analogue of Theorem 2.1:

THEOREM 2.2. *Assume that E_h is consistent with the parabolic operator $E(t)=E(t;P)$. Then E_h is parabolic if and only if for any Q_h consistent with a differential operator $Q(D)$ of order q, any $u_0 \in L^2$, any $t \geqq 0$ which is >0 if $q>0$, and any pair of sequences $\{h_j\}_{j=1}^{\infty}$, $\{n_j\}_{j=1}^{\infty}$ with $h_j \to 0$, $n_j k_j \to t$, as $j \to \infty$ we have*

$$(2.15) \qquad \|Q_{h_j} E_{h_j}^{n_j} u_0 - Q(D)E(t)u_0\| \to 0, \quad j \to \infty .$$

PROOF. Assume first that (2.15) holds. We can then prove in the same way as in the proof of Theorem 2.1, using Banach–Steinhaus' theorem that (2.14) holds so that E_h is parabolic. On the other hand, if E_h is parabolic, the relation (2.15) follows as in the proof of Theorem 2.1 from the following lemma:

LEMMA 2.6. *If E_h is consistent with $E(t;P)$ and Q_h is consistent with $Q(D)$, then*

$$Q_{h_j}(\xi)\, E_{h_j}(\xi)^{n_j} \to Q(\xi) \exp(tP(\xi)), \quad h_j \to 0,\ n_j k_j \to t ,$$

uniformly on compact sets in ξ.

PROOF. Follows from Lemma 2.2 since by Lemma 2.5, $Q_{h_j}(\xi) \to Q(\xi)$ when $j \to \infty$.

Also parabolicity can be expressed in terms of symbols:

LEMMA 2.7. *The operator E_h is parabolic if and only if for any Q_h consistent with a differential operator $Q(D)$ of order q and any τ, T with $\tau > 0$,*

$$(2.16) \qquad \sup\{|Q_h(\xi)E_h(\xi)^n|;\ q\tau \leqq nk \leqq T,\ \xi \text{ real}\} < \infty .$$

PROOF. Analogous to the proof of Lemma 2.3.

We have the following consequence:

LEMMA 2.8. *If E_h is parabolic and consistent with $E(t;P)$, then $E(t;P)$ is parabolic.*

PROOF. Follows at once from Lemmas 1.3, 2.6, and 2.7.

We want to study the rate of convergence in (2.15) and introduce the following definitions: Let E_h be consistent with $E(t)=E(t;P)$. We say that the order of accuracy is m if for any infinitely differentiable solution $u(x,t)$ of (1.1),

$$A_h u(x,t+k) = B_h u(x,t) + O(h^{p+m}), \quad h \to 0 .$$

Also, let Q_h be consistent with the differential operator $Q(D)$. We then say that the order of accuracy is m if for any infinitely differentiable function $u(x)$,

$$Q_h u(x) = Q(D)\, u(x) + O(h^m), \quad h \to 0 .$$

These conditions can also be expressed in terms of the symbols:

LEMMA 2.9. *Let E_h and Q_h be consistent with $E(t;P)$ and $Q(D)$, respectively. The order of accuracy is m, respectively, if (q is the order of $Q(D)$),*

$$E_h(h^{-1}\xi) = \exp\big(kP(h^{-1}\xi) + O(h^{p+m} + |\xi|^{p+m})\big), \quad h,\xi \to 0 ,$$
$$h^q Q_h(h^{-1}\xi) = h^q Q(h^{-1}\xi) + O(h^{q+m} + |\xi|^{q+m}), \quad h,\xi \to 0 .$$

PROOF. Analogous to the proof of Lemma 2.1.

LEMMA 2.10. *Assume that E_h and Q_h are consistent with $E(t;P)$ and $Q(D)$, respectively, and that both have order of accuracy m. Then if both $E(t;P)$ and E_h are parabolic, there is for any τ, T with $\tau > 0$ a constant C such that for $q\tau \leqq nk \leqq T$ (q is the order of $Q(D)$),*

$$(2.17) \qquad |Q_h(\xi)\, E_h(\xi)^n - Q(\xi) \exp\big(nkP(\xi)\big)| \leqq C h^m (1 + |\xi|^{p+m}) .$$

PROOF. Since the orders of accuracy of E_h and Q_h are both m, we obtain easily by Lemma 2.9 that for $h \leqq h_0$, and all real ξ,

$$(2.18) \qquad |E_h(\xi) - \exp\big(kP(\xi)\big)| \leqq C h^{p+m} (1 + |\xi|^{p+m}) ,$$

$$(2.19) \qquad |Q_h(\xi) - Q(\xi)| \leqq C h^m (1 + |\xi|^{q+m}) ,$$

and we also have for $h \leqq h_0$ and ξ real,

$$(2.20) \qquad |Q_h(\xi)| \leqq C(1 + |\xi|^q) .$$

We can write

$$(2.21) \qquad Q_h(\xi)\, E_h(\xi)^n - Q(\xi) \exp\big(nkP(\xi)\big)$$
$$= \sum_{j=0}^{n-1} Q_h(\xi)\, E_h(\xi)^j \,[E_h(\xi) - \exp\big(kP(\xi)\big)]\, \exp\big((n-1-j)kP(\xi)\big) + $$
$$+ [Q_h(\xi) - Q(\xi)] \exp\big(nkP(\xi)\big) .$$

We want to estimate the different terms on the right in (2.21). Consider first the terms in the sum with $\frac{1}{2}n \leqq j < n$. Then $Q_h(\xi) E_h(\xi)^j$ is bounded since E_h is parabolic and $\frac{1}{2}q\tau \leqq jk \leqq T$, $\exp\big((n-1-j)kP(\xi)\big)$ is bounded since $E(t;P)$ is uniformly bounded when $0 \leqq t \leqq T$, and thus by (2.18) we have for these terms

$$(2.22) \qquad |Q_h(\xi)\, E_h(\xi)^j\, [E_h(\xi) - \exp\big(kP(\xi)\big)] \exp\big((n-1-j)kP(\xi)\big)|$$
$$\leqq C h^{p+m} (1 + |\xi|^{p+m}) .$$

Consider now the terms in the sum with $0 \leqq j < \frac{1}{2}n$. Then $E_h(\xi)^j$ is bounded since E_h is stable and $Q_h(\xi)\exp((n-1-j)kP(\xi))$ is bounded by (2.20) and since $E(t;P)$ is parabolic and $\frac{1}{3}q\tau \leqq (n-1-j)k \leqq T$ for small h. Therefore, by (2.18) the estimate (2.22) holds also for these terms. Finally, for the last term in (2.21) we have by (2.19) since $E(t;P)$ is parabolic and $q\tau \leqq nk \leqq T$,

$$|[Q_h(\xi)-Q(\xi)]\exp(nkP(\xi))| \leqq Ch^m .$$

Notice that if $q=0$, we have by our conventions that $Q_h(\xi)-Q(\xi)=0$. Altogether, we get since $k/h^p=\lambda$ and $nk \leqq T$,

$$\begin{aligned} |Q_h(\xi)\,E_h(\xi)^n - Q(\xi)\exp(nkP(\xi))| &\leqq C\left[\sum_{j=0}^{n-1} h^{p+m}(1+|\xi|^{p+m})+h^m\right] \\ &\leqq Ch^m(1+|\xi|^{p+m}) , \end{aligned}$$

which proves the lemma.

To be able to state the result on the rate of convergence we introduce the Hilbert space H_s of functions $u \in L^2$ such that $(1+|\xi|^2)^{\frac{1}{2}s}\hat{u}(\xi) \in L^2$ with the norm

$$\|u\|_s = \left(\int (1+|\xi|^2)^s\,|\hat{u}(\xi)|^2\,d\xi\right)^{\frac{1}{2}},$$

where $\hat{u}$ is the Plancherel–Fourier transform on L^2. We recall Sobolev's inequality: if $\tilde{d} > \frac{1}{2}d$ there is a constant C such that for $u \in H_{\tilde{d}}$,

(2.23) $$\sup_x |u(x)| \leqq C\|u\|_{\tilde{d}} .$$

The result is then:

Theorem 2.3. *Assume that E_h and Q_h are consistent with $E(t;P)$ and $Q(D)$, respectively, and that both have order of accuracy m. Then, if both $E(t;P)$ and E_h are parabolic, there are for each τ, T with $\tau > 0$ constants C_1, C_2 such that for $q\tau \leqq nk \leqq T$ (q is the order of $Q(D)$),*

(2.24) $$\|Q_h E_h^n u_0 - Q(D)E(nk)u_0\| \leqq C_1 h^m \|u_0\|_{p+m} ,$$

(2.25) $$\sup_x |Q_h E_h^n u_0(x) - Q(D)E(nk)u_0(x)| \leqq C_2 h^m \|u_0\|_{p+m+\tilde{d}} ,$$

for $u_0 \in H_{p+m}$ and $u_0 \in H_{p+m+\tilde{d}}$ ($\tilde{d} > \frac{1}{2}d$), respectively.

Proof. Taking Fourier transforms and using Parseval's relation (2.2,4) follows at once from Lemma 2.10. Multiplying (2.17) by $(1+|\xi|^2)^{\frac{1}{2}\tilde{d}}$ we obtain in the same manner for $u_0 \in H_{p+m+\tilde{d}}$,

$$\|Q_h E_h^n u_0 - Q(D)E(nk)u_0\|_{\tilde{d}} \leqq Ch^m \|u_0\|_{p+m+\tilde{d}} ,$$

and (2.25) then follows by (2.23).

3. Algebraic conditions for parabolicity.

We shall now discuss necessary and sufficient conditions for parabolicity of the operator E_h. It turns out that in addition to the conditions for stability, the conditions for parabolicity can be expressed in terms of properties of the spectral radius $\varrho_h(\xi)$ of the matrix $E_h(\xi)$.

According to Lemma 2.3, the stability of E_h is equivalent to

$$\sup\{|E_h(\xi)^n|;\ 0 \leqq nk \leqq T,\ \xi \text{ real}\} < \infty \tag{3.1}$$

for all $T > 0$. It is well known that the von Neumann condition

$$\varrho_h(\xi) \leqq 1 + Ck, \qquad \xi \text{ real},\ h \leqq h_0\,, \tag{3.2}$$

is a necessary condition for (3.1). Necessary and sufficient algebraic conditions for (3.1) have been given by Kreiss [10] and Buchanan [3]. We will quote below one version of Kreiss' theorem.

Let H be a positive definite $N \times N$ matrix. Then if u is a complex N-vector,

$$|u|_H = (Hu,u)^{\frac{1}{2}}\,,$$

where $(u,v) = \sum_{j=1}^N u_j \bar{v}_j$ is the ordinary scalar product, defines a norm, equivalent to $|u|$. The corresponding matrix norm is

$$|A|_H = \sup_{u \neq 0} |Au|_H/|u|_H = \sup_{u \neq 0} [(A^*HAu,u)/(Hu,u)]^{\frac{1}{2}}\,,$$

and if we generally let $\varrho(A)$ be the spectral radius of A, we have for any such H,

$$\varrho(A) \leqq |A|_H\,.$$

We then have the following discrete analogue of Lemma 1.2:

Lemma 3.1 *Let $\mathscr{F}$ denote a family of $N \times N$ matrices. Then*

$$\sup\{|A^n|;\ A \in \mathscr{F},\ n = 0,1,\ldots\} < \infty \tag{3.3}$$

if there is a positive constant C and for any $A \in \mathscr{F}$ a hermitean matrix H with

$$C^{-1}I \leqq H \leqq CI\,, \tag{3.4}$$

and

$$|A|_H \leqq 1\,.$$

On the other hand, if (3.3) holds, then $\varrho(A) \leqq 1$ for any $A \in \mathscr{F}$ and if $0 \leqq \gamma < 1$ there is a constant C, and for any $A \in \mathscr{F}$ a hermitean matrix H satisfying (3.4) and

$$|A|_H \leqq \gamma\varrho(A) + 1 - \gamma\,.$$

PROOF. With $\gamma=0$ in the second part, the lemma is proved in [10]. The above modification was discovered by Widlund [16] and is easily proved by reviewing the proof in [10].

From Lemma 3.1 we easily obtain:

THEOREM 3.1. *The operator E_h is stable if there are positive constants C_1, C_2 and for real ξ and $h \leqq h_0$ a hermitean matrix $H_h(\xi)$ such that for these ξ and h,*

$$C_1^{-1} I \leqq H_h(\xi) \leqq C_1 I \tag{3.5}$$

and

$$|E_h(\xi)|_{H_h(\xi)} \leqq 1 + C_2 k \, .$$

On the other hand if E_h is stable and $\varrho_h(\xi)=\varrho(E_h(\xi))$ we have (3.2) *and if* $0 \leqq \gamma < 1$ *there are positive constants C_1, C_2 and for each real ξ and $h \leqq h_0$ a hermitean matrix $H_h(\xi)$ satisfying* (3.5) *and*

$$|E_h(\xi)|_{H_h(\xi)} \leqq \gamma \varrho_h(\xi) + (1-\gamma) + C_2 k \, . \tag{3.6}$$

An important tool in the sequel is the following lemma, which can be considered as a discrete analogue of Lemma 1.4.

LEMMA 3.2. *There exists a constant C_N depending only on N such that for any $N \times N$ matrix A with spectral radius ϱ we have for $n \geqq N$,*

$$|A^n| \leqq C_N \varrho^{n-N+1} \left[\varrho^{N-1} + (n|A-I|)^{N-1}\right] . \tag{3.7}$$

PROOF. Beside the euclidean matrix norm in which the lemma is expressed we introduce the equivalent norm

$$|A|_{\max} = \max_j \sum_{k=1}^{N} |a_{jk}|, \qquad A = (a_{jk}) \, ,$$

which corresponds to the vector norm

$$|u|_{\max} = \max_j |u_j|, \qquad u = (u_1, \ldots, u_N) \, .$$

We have

$$|u|_{\max} \leqq |u| \leqq N^{\frac{1}{2}} |u|_{\max} \, ,$$

and it follows that

$$|A| \leqq N^{\frac{1}{2}} |A|_{\max} \leqq N |A| \, .$$

For any matrix $A=(a_{jk})$ let $A_{\mathrm{abs}}=(|a_{jk}|)$. Then $|A|_{\max}=|A_{\mathrm{abs}}|_{\max}$.

We now turn to (3.7). By Schur's theorem any square matrix is unitarily equivalent to an upper triangular matrix. We can therefore without restricting the generality assume that A has the form

$$A = D + F \, ,$$

where D is a diagonal matrix and F is purely upper triangular. We have $(F_{\text{abs}})^N = 0$ and

$$|F|_{\max} \leqq |F+D-I|_{\max} = |A-I|_{\max} .$$

Therefore, for $n \geqq N$,

$$\begin{aligned}
|A^n| \leqq N^{\frac{1}{2}} |(A^n)_{\text{abs}}|_{\max} &\leqq N^{\frac{1}{2}} |(A_{\text{abs}})^n|_{\max} \\
&\leqq N^{\frac{1}{2}} |(\varrho I + F_{\text{abs}})^n|_{\max} \\
&\leqq N^{\frac{1}{2}} \sum_{j=0}^{N-1} \binom{n}{j} \varrho^{n-j} |F_{\text{abs}}|_{\max}^j \\
&\leqq N^{\frac{1}{2}} \varrho^{n-N+1} \sum_{j=0}^{N-1} \binom{n}{j} \varrho^{N-1-j} |A-I|_{\max}^j \\
&\leqq N^{\frac{1}{2}N} \varrho^{n-N+1} \sum_{j=0}^{N-1} \binom{n}{j} \varrho^{N-1-j} |A-I|^j \\
&\leqq C_N \varrho^{n-N+1} [\varrho^{N-1} + (n|A-I|)^{N-1}] .
\end{aligned}$$

This proves the lemma.

We will also need the Seidenberg–Tarski elimination theorem:

Lemma 3.3. *Let $S = S(\sigma, \tau)$ be a finite system of polynomial equations and inequalities in $\sigma = (\sigma_1, \ldots, \sigma_\alpha)$ and $\tau = (\tau_1, \ldots, \tau_\beta)$ with real coefficients:*

$$\begin{aligned}
S(\sigma,\tau): \quad p_k(\sigma,\tau) &= 0, \qquad k = 1,\ldots,\gamma , \\
q_k(\sigma,\tau) &\leqq 0, \qquad k = 1,\ldots,\delta .
\end{aligned}$$

Then there exists a finite set $\Sigma_1(\sigma), \ldots, \Sigma_\mu(\sigma)$ where $\Sigma_j(\sigma)$ is a finite system of polynomial equations and inequalities in σ with real coefficients:

$$\begin{aligned}
\Sigma_j(\sigma): \quad p_{jk}(\sigma) &= 0, \qquad k = 1,\ldots,\gamma_j , \\
q_{jk}(\sigma) &\leqq 0, \qquad k = 1,\ldots,\delta_j ,
\end{aligned}$$

such that for any given σ, the system $S(\sigma,\tau)$ has a solution (σ,τ) if and only if at least one of the systems $\Sigma_j(\sigma)$, $j = 1,\ldots,\mu$, is satisfied by σ.

Proof. See e.g. [5, p. 225].

We can now state the main result in this section:

Theorem 3.2. *Assume that the operator E_h is consistent with some $E(t;P)$. Then E_h is parabolic if and only if it is stable and if there are positive constants C_1, C_2, h_0, and ν such that the spectral radius $\varrho_h(\xi)$ of $E_h(\xi)$ satisfies*

$$\varrho_h(\xi) \leqq 1 - C_1 k |\xi|^\nu + C_2 k, \qquad h|\xi_j| \leqq \pi, \ h \leqq h_0 . \tag{3.8}$$

Proof. We first prove the sufficiency of this condition for parabolicity. By Lemma 2.6. we have to prove the boundedness of $Q_h(\xi)E_h(\xi)^n$ if Q_h is consistent with a differential operator $Q(D)$ of order q and $q\tau \leqq nk \leqq T$, where $\tau > 0$. Since we assume stability, we only have to prove this for $q>0$ and since $Q_h(\xi)$ and $E_h(\xi)$ are both periodic with period $2\pi/h$ we only have to consider $h|\xi_j| \leqq \pi$. By Lemmas 2.1 and 2.5 the consistency of E_h with $E(t;P)$ and Q_h with $Q(D)$ implies for $h \leqq h_0$,

$$\begin{aligned} |E_h(\xi)-I| &\leqq Ck(1+|\xi|^p) , \\ |Q_h(\xi)| &\leqq C(1+|\xi|^q) . \end{aligned} \tag{3.9}$$

We therefore obtain by Lemma 3.2 for $0<\tau \leqq nk \leqq T$, $n \geqq N$, if we use the uniform boundedness of $\varrho_h(\xi)$ for real ξ,

$$\begin{aligned} |Q_h(\xi)E_h(\xi)^n| &\leqq C(1+|\xi|^q)\, \varrho_h(\xi)^{n-N+1}\, [\varrho_h(\xi)^{N-1} + (n\,|E_h(\xi)-I|)^{N-1}] \\ &\leqq C(1+|\xi|^{q+(N-1)p})\, \varrho_h(\xi)^{n-N+1} . \end{aligned}$$

By condition (3.8) we obtain for $h|\xi_j| \leqq \pi$,

$$\begin{aligned} |Q_h(\xi)E_h(\xi)^n| &\leqq C(1+|\xi|^{q+(N-1)p}) \exp(-nkC_1|\xi|^\nu + nkC_2) \\ &\leqq C(1+|\xi|^{q+(N-1)p}) \exp(-\tau C_1|\xi|^\nu + TC_2) \leqq C , \end{aligned}$$

since $\nu > 0$. This proves the sufficiency part of the theorem.

We now turn to the necessity part of the theorem. We introduce the operators $\partial_{j,h}$ with symbols

$$\partial_{j,h}(\xi) = (ih)^{-1}(\exp(ih\xi_j)-1) = h^{-1}[\sin h\xi_j + i(1-\cos h\xi_j)] ,$$

which are consistent with $\partial_j = i^{-1}\partial/\partial x_j$, $j=1,\ldots,d$. We set

$$\partial_h(\xi) = (\partial_{1,h}(\xi),\ldots,\partial_{d,h}(\xi))$$

so that

$$|\partial_h(\xi)| = \left(\sum_{j=1}^d |\partial_{j,h}(\xi)|^2\right)^{\frac12} = h^{-1}\left[2\sum_{j=1}^d (1-\cos h\xi_j)\right]^{\frac12} .$$

By the definition of parabolicity of E_h we have for some constant C_3 and $h \leqq h_0$,

$$(1+|\partial_h(\xi)|)\, \varrho_h(\xi)^n \leqq \exp(C_3), \qquad nk=1 ,$$

or

$$\varrho_h(\xi) \leqq \exp(C_3 k) \exp[-k \log(1+|\partial_h(\xi)|)] .$$

Since for $0<k \leqq k_0 = \lambda h_0^p$,

$$\begin{aligned} \exp(C_3 k) &\leqq 1 + C_4 k , \\ \exp[-k \log(1+|\partial_h(\xi)|)] &\leqq 1 - C_5 k \log(1+|\partial_h(\xi)|) , \end{aligned}$$

where

$$C_4 = C_2 \exp(C_2), \qquad C_5 = \exp(-\sup_{0<k\leq k_0} \{k \log(1+h^{-1}2d^{\frac{1}{2}})\}) ,$$

we get for $0<k\leqq k_0$,

$$\varrho_h(\xi) \leqq 1+C_4k-C_5k\log(1+|\partial_h(\xi)|) . \tag{3.10}$$

We introduce for $r\geqq 0$ the function

$$\varphi(r) = \sup\{k^{-1}[\varrho_h(\xi)-1];\ \xi \text{ real},\ 0<h\leqq h_0,\ |\partial_h(\xi)|=r\} . \tag{3.11}$$

This function enjoys the following properties:

i) $\varphi(r)\leqq C_4-C_5\log(1+r)$,
ii) $\varphi(r)$ is an algebraic function of r for large r.

The first condition follows at once from (3.10). To prove the second, we introduce in $E_h(\xi)$ for $h\leqq h_0$ new real variables $s=(s_1,\ldots,s_d)$, $c=(c_1,\ldots,c_d)$ by

$$s_j = s_{j,h}(\xi) = h^{-1}\sin h\xi_j ,$$
$$c_j = c_{j,h}(\xi) = h^{-2}2(1-\cos h\xi_j) ,$$

that is we substitute in $A_h(\xi)$, $B_h(\xi)$,

$$\exp(\pm ih\xi_j) = 1\pm ihs_j-\tfrac{1}{2}h^2c_j ,$$

and obtain for $A_h(\xi)$ and $B_h(\xi)$ polynomials $A(h,s,c)$ and $B(h,s,c)$. Between s_j and c_j we have the relation

$$s_j^2 = c_j-\tfrac{1}{4}h^2c_j^2, \qquad j = 1,\ldots,d , \tag{3.12}$$

and if we agree to use this relation to replace s_j^2 whenever possible, the representations $A(h,s,c)$ and $B(h,s,c)$ become unique. We set

$$E(h,s,c) = A(h,s,c)^{-1}B(h,s,c)$$

for $h\leqq h_0$ and s,c satisfying (3.12). Because of the consistency we can conclude that

$$E(h,s,c) = I+k\tilde{P}(h,s,c) ,$$

where $\tilde{P}$ is analytic for the h,s,c in question; we have indeed

$$\tilde{P}(0,s_h(h^{-1}\xi),c_h(h^{-1}\xi)) = P(h^{-1}\xi)+o(1+|h^{-1}\xi|^p), \qquad h,\xi\to 0 .$$

If we introduce the spectral radius $\varrho(h,s,c)$ of $E(h,s,c)$ and the function

$$\psi(h,s,c) = k^{-1}(\varrho(h,s,c)-1) ,$$

then the consistency proves that $\psi(h,s,c)$ can be defined to be continuous for $0\leqq h\leqq h_0$ and s,c satisfying (3.12), and the definition (3.11) can also be written

(3.13)

$$\varphi(r) = \sup\{\psi(h,s,c);\ 0 \leqq h \leqq h_0,\ 4s_j^2 + h^2c_j^2 = 4c_j,\ j = 1,\ldots,d,\ \textstyle\sum_1^d c_j = r^2\}.$$

Assume now that $r \geqq 0$, $\varphi = \varphi(r)$. Then since the supremum in (3.13) is over a compact set in the (h,s,c)-space, there are h,s,c satisfying $\varphi = \psi(h,s,c)$ and the side-conditions. By the definition of $\psi(h,s,c)$ this means that there is an eigenvalue $\varkappa = \varkappa_1 + i\varkappa_2$ of $\tilde{P}(h,s,c)$ such that

$$k^{-1}(|1+k\varkappa|-1) = \varphi\ .$$

Altogether this means that the following system $S = S(r,\varphi,h,k,s,c,\varkappa_1,\varkappa_2)$ of polynomial equations and inequalities with real coefficients is satisfied:

$$\begin{aligned} S:\quad & k - \lambda h^p = 0\ ,\\ & \operatorname{Re}\det\{A(h,s,c) - (1+k(\varkappa_1+i\varkappa_2))B(h,s,c)\} = 0\ ,\\ & \operatorname{Im}\det\{A(h,s,c) - (1+k(\varkappa_1+i\varkappa_2))B(h,s,c)\} = 0\ ,\\ & 2\varkappa_1 + k(\varkappa_1^2+\varkappa_2^2) - 2\varphi - k\varphi^2 = 0\ ,\\ & 4s_j^2 + h^2c_j^2 - 4c_j = 0, \quad j = 1,\ldots,d\ ,\\ & \sum_{j=1}^{d} c_j - r^2 = 0\ ,\\ & -h \leqq 0, \qquad h - h_0 \leqq 0\ . \end{aligned}$$

Let $\Sigma_1(r,\varphi),\ldots,\Sigma_\mu(r,\varphi)$ be a finite set of finite systems of polynomial equations and inequalities in r and φ with real coefficients which corresponds to S by the Seidenberg–Tarski theorem after elimination of $(h,k,s,c,\varkappa_1,\varkappa_2)$. Then at least one of these systems $\Sigma_\tau(r,\varphi)$ is satisfied for $\varphi = \varphi(r)$. At least one equation has to occur in Σ_τ for $\varphi = \varphi(r)$, for if this were not the case Σ_τ would also be satisfied by (r,Φ) with $\Phi > \varphi(r)$ and sufficiently close to $\varphi(r)$. But then by the converse part of the Seidenberg–Tarski theorem we would be able to find $(\tilde{h},\tilde{k},\tilde{s},\tilde{c},\tilde{\varkappa}_1,\tilde{\varkappa}_2)$ such that $(r,\Phi,\tilde{h},\tilde{k},\tilde{s},\tilde{c},\tilde{\varkappa}_1,\tilde{\varkappa}_2)$ satisfies S. But then $\tilde{P}$ would have an eigenvalue $\tilde{\varkappa}$ with

$$k^{-1}(|1+k\tilde{\varkappa}|-1) = \Phi > \varphi$$

for some h,s,c satisfying the sideconditions, and this would contradict the definition of $\varphi(r)$ as the supremum in (3.13). Let $F(r,\varphi)$ be the product of all the polynomials occuring in all the $\Sigma_\tau(r,\varphi)$. Then $F(r,\varphi)$ is a polynomial and $F(r,\varphi(r)) \equiv 0$, and we can finally conclude that $\varphi(r)$ is algebraic for large r.

Since $\varphi(r)$ is an algebraic function of r for large r we have by a Puisseux development of $\varphi(r)$ around $r = \infty$, with $C_6 \neq 0, \nu$ rational,

$$\varphi(r) = -2C_6 r^\nu(1+o(1)), \quad r \to \infty\ .$$

Condition i) gives $C_6 > 0$, $\nu > 0$, and we conclude that there is a r_0 such that

$$\varphi(r) \leqq -C_6 r^\nu, \qquad r \geqq r_0 .$$

Since by i) $\varphi(r) - C_4 \leqq 0$ for all $r \geqq 0$ we have with $C_2 = C_6 r_0^\nu + C_4$,

$$\varphi(r) \leqq -C_6 r^\nu + C_2, \qquad r \geqq 0 .$$

We therefore have, using (3.11), for any real ξ, and $0 < h \leqq h_0$,

$$k^{-1}[\varrho_h(\xi) - 1] \leqq \varphi(|\partial_h(\xi)|) \leqq -C_6 |\partial_h(\xi)|^\nu + C_2 ,$$

or in view of the trivial estimate $|\partial_h(\xi)| \geqq (2/\pi)|\xi|$, with $C_1 = (2/\pi)^\nu C_6$,

$$\varrho_h(\xi) \leqq 1 - C_1 k |\xi|^\nu + C_2 k .$$

This proves (3.8) and so concludes the proof of the theorem.

It follows from the proof of Theorem 3.2 that condition (3.8) is equivalent to

$$\sup\{|Q_h(\xi) E_h(\xi)^n| ;\ \tau \leqq nk \leqq T,\ \xi \text{ real}\} < \infty$$

for all τ, T with $0 < \tau < T$, and all Q_h consistent with a differential operator $Q(D)$. In the case where the matrix $E_h(\xi)$ is normal, in particular in the scalar case ($N = 1$), the condition (3.8) is sufficient for parabolicity of E_h; in this case we automatically have stability since for $0 \leqq nk \leqq T$, $h|\xi_j| \leqq \pi$,

$$|E_h(\xi)^n| = \varrho_h(\xi)^n \leqq \exp[(-C_1|\xi|^\nu + C_2) nk] \leqq \exp(C_2 T) .$$

Also, as will be proved in Lemma 3.4 below, the condition (3.8) with $\nu = p$ is sufficient for parabolicity. In the general case, however, the stability of E_h has to be explicitely assumed. To see this, we consider an example. Let

$$\sigma(\xi) = \tfrac{1}{2}(1 - \cos\xi) = \sin^2 \tfrac{1}{2}\xi ,$$
$$a_h(\xi) = [1 + i\sigma(\xi)^2 - \sigma(\xi)^4]\,[1 - h^2 \sigma(\xi)] ,$$

and let E_h be the operator with symbol ($N = 2$, $d = 1$)

$$E_h(\xi) = \begin{pmatrix} a_h(h\xi) & \sigma(h\xi)^3 \\ 0 & a_h(h\xi) \end{pmatrix} = a_h(h\xi) I + \sigma(h\xi)^3 J . \tag{3.14}$$

We find

$$E_h(h^{-1}\xi) = (1 - \tfrac{1}{4}h^2\xi^2 + \tfrac{1}{16} i\xi^4) I + o(h^4 + \xi^4), \qquad h, \xi \to 0 ,$$

so that E_h is consistent with the parabolic operator $E(t; P)$ where

$$P(\xi) = (-\tfrac{1}{4}\xi^2 + \tfrac{1}{16} i\xi^4) I, \qquad \lambda = 1,\ p = 4,\ \mu = 2 .$$

Clearly

$$\varrho_h(\xi) = |a_h(h\xi)| \leqq 1-h^2\sin^2\tfrac{1}{2}h\xi \leqq 1-\pi^{-2}k\xi^2, \qquad |\xi|h \leqq \pi ,$$

and so condition (3.8) is satisfied. On the other hand if $h=n^{-\frac{1}{2}}$, and

$$\xi_h = h^{-1}\arccos(1-2h^{\frac{1}{2}}) ,$$

we have $\sigma(h\xi_h)=h^{\frac{1}{2}}$, $a_h(h\xi_h) \to 1$, and $|a_h(h\xi_h)^n| \to e^{-\frac{1}{2}}$ as $n \to \infty$, and so

$$E_h(\xi_h)^n = a_h(h\xi_h)^n I + n a_h(h\xi_h)^{n-1}\sigma(h\xi_h)^3 J$$

is not bounded as $n \to \infty$. Since $kn=h^2 \leqq 1$, this proves that E_h is not stable. We notice for later use that ξ_h is chosen so that $h\xi_h \to 0$ as $n \to \infty$.

By the proof of Theorem 3.2, if E_h is parabolic there is a largest $\nu > 0$ such that (3.8) holds. We call this largest ν the order of parabolicity of E_h.

We say that E_h is locally stable if there is a $\gamma > 0$ such that for any $T > 0$,

$$\sup\{|E_h(\xi)^n|;\ h|\xi| \leqq \gamma,\ 0 \leqq nk \leqq T\} < \infty . \tag{3.15}$$

Similarly, we say that E_h is locally parabolic if there is a $\gamma > 0$ such that for any τ, T with $\tau > 0$, and any Q_h consistent with a differential operator $Q(D)$ of order $q \geqq 0$,

$$\sup\{|Q_h(\xi)E_h(\xi)^n|;\ h|\xi| \leqq \gamma,\ q\tau \leqq nk \leqq T\} < \infty . \tag{3.16}$$

It follows as in the proof of Theorem 3.2 that E_h is locally parabolic if and only if it is locally stable and if there are positive constants C_1, C_2, γ, ν such that

$$\varrho_h(\xi) \leqq 1-C_1k|\xi|^\nu+C_2k, \qquad h|\xi| \leqq \gamma . \tag{3.17}$$

It also follows as in the proof of Theorem 3.2 that there is a largest ν for which this is possible. This largest ν we then call the order of local parabolicity. Notice that this definition does not demand that E_h is parabolic; E_h does not have to be stable and (3.8) does not have to hold for $\gamma < |\xi|h$, $|\xi_j|h \leqq \pi$.

The above example (3.14) proves that (3.17) is not enough for local parabolicity; in general we have to assume explicitely local stability. The normal case is again an exception. We also have the following:

LEMMA 3.4. *If E_h is consistent with $E(t;P)$ and if for some positive constants C_1, C_2, γ,*

$$\varrho_h(\xi) \leqq 1-C_1k|\xi|^p+C_2k , \tag{3.18}$$

for $h|\xi| \leqq \gamma$, then E_h is locally parabolic of order p. If (3.18) *holds for $h|\xi_j| \leqq \pi$, then E_h is parabolic of order p.*

Proof. We have by Lemma 3.2, (3.9), and (3.18) for $h|\xi| \leqq \gamma$ ($h|\xi_j| \leqq \pi$), $0 \leqq nk \leqq T$,

$$\begin{aligned}|E_h(\xi)^n| &\leqq C[1+(nk(1+|\xi|^p))^{N-1}] \exp(-C_1 nk|\xi|^p + C_2 nk) \\ &\leqq C[1+(nk|\xi|^p)^{N-1}] \exp(-C_1 nk|\xi|^p) \leqq C ,\end{aligned}$$

which proves the lemma.

We have the following sharpening of Lemma 2.8:

Lemma 3.5. *If E_h is locally parabolic of order ν, and consistent with $E(t;P)$, then $E(t;P)$ is parabolic of order at least ν.*

Proof. We first prove that the initial-value problem (1.1), (1.2) is correctly posed. This follows at once from Lemma 2.2 and (3.15) since

$$|\exp(tP(\xi))| = \lim_{\substack{nk=t \\ h\to 0}} |E_h(\xi)^n| \leqq C .$$

By the above definition we have for $h|\xi| \leqq \gamma$,

$$\varrho_h(\xi) \leqq 1 - C_1 k|\xi|^\nu + C_2 k ,$$

and so for $nk=1$ if h is so small that $h|\xi| \leqq \gamma$,

$$\varrho_h(\xi)^n \leqq (1 - C_1 k|\xi|^\nu + C_2 k)^n \leqq \exp(-C_1|\xi|^\nu + C_2) .$$

It follows from Lemmas 1.3, 2.2, and 3.2, (3.9) and (3.17),

$$\begin{aligned}\exp(\Lambda(\xi)) \leqq |\exp(P(\xi))| &= \lim_{\substack{nk=1 \\ h\to 0}} |E_h(\xi)^n| \\ &\leqq \overline{\lim_{\substack{nk=1 \\ h\to 0}}}\, C(1+|\xi|^p)^{N-1} \varrho_h(\xi)^n \leqq \exp(-C_3|\xi|^\nu + C_4) ,\end{aligned}$$

which concludes the proof of the lemma.

In practice, the problem of constructing a parabolic E_h consistent with a parabolic $E(t;P)$ is essentially solved if we have found a locally parabolic E_h:

Lemma 3.6. *Assume that E_h is consistent with $E(t;P)$ and of order of accuracy m, where P has order p. Assume further that E_h is locally parabolic of order ν. Let $\sigma(\xi)$ be the trigonometric polynomial*

$$\sigma(\xi) = 1 - d^{-1} 2^{-\varkappa} \sum_{j=1}^{d} (1 - \cos \xi_j)^\varkappa ,$$

where $2\varkappa \geqq p+m$. Then

$$E_h'(\xi) = E_h(\xi)\, \sigma(h\xi)^s ,$$

where s is a natural number defines an operator E_h' which is also consistent with $E(t;P)$ and has order or accuracy m. Further, if s is large enough, E_h' is parabolic of order ν. Finally E_h' is explicit with E_h.

PROOF. Clearly E_h' is also consistent with $E(t;P)$ and with the same order of accuracy as E_h since for any natural number s,

$$\sigma(\xi)^s = 1+O(|\xi|^{2\varkappa}) = 1+O(|\xi|^{p+m}), \quad \xi \to 0 .$$

By assumption there is a $\gamma > 0$ such that (3.16) holds. The result then easily follows if we choose s so large that

$$\sup\{|E_h(\xi)|\sigma(h\xi)^s;\ \gamma \leqq h|\xi|,\ h|\xi_j| \leqq \pi,\ h \leqq h_0\} < 1 ,$$

which is clearly possible.

We shall look a little closer at operators which are parabolic in Petrowsky's sense. We have

THEOREM 3.3. *Assume that the operator E_h is consistent with $E(t;P)$ which is parabolic of order p where p is the order of the differential operator $P(D)$. Then E_h is locally parabolic of order p and E_h is parabolic of order p if and only if for any $\gamma > 0$, there is a $h_0 > 0$ such that*

$$\sup\{\varrho_h(\xi);\ \gamma \leqq h|\xi|,\ h|\xi_j| \leqq \pi,\ h \leqq h_0\} < 1 . \tag{3.19}$$

PROOF. If $E(t;P)$ is parabolic in Petrowsky's sense, we have

$$\Lambda(\xi) \leqq -C_1|\xi|^p + C_2 .$$

Since E_h is consistent with $E(t;P)$ we have by Lemma 2.1,

$$E_h(h^{-1}\xi) = \exp(kP(h^{-1}\xi)+o(k+|\xi|^p)), \quad k,\xi \to 0 ,$$

and so

$$\begin{aligned}\varrho_h(h^{-1}\xi) &\leqq \exp(-C_1k|h^{-1}\xi|^p+C_2k+o(k+|\xi|^p)) \\ &\leqq 1-C_3|\xi|^p+C_4k, \qquad \text{for } |\xi| \leqq \gamma,\ h \leqq h_0 ,\end{aligned}$$

if γ and h_0 are small enough. By Lemma 3.4 this proves that E_h is locally parabolic of order p. Clearly, if condition (3.19) holds we have for any Q_h consistent with a differential operator $Q(D)$ of order $q \geqq 0$ as in the proof of Theorem 3.2 for any τ, T, with $\tau > 0$,

$$\sup\{|Q_h(\xi)E_h(\xi)^n|;\ \gamma < h|\xi|,\ h|\xi_j| \leqq \pi,\ h \leqq h_0,\ q\tau \leqq nk \leqq T\} < \infty .$$

Since the condition (3.19) is obviously necessary for parabolicity of order p, this completes the proof of the theorem.

If $E(t;P)$ is parabolic of order $\mu < p$ we shall see that the situation is more complicated; we are going to give examples of explicit operators E_h, consistent with scalar parabolic operators $E(t;P)$ and which are

i) not stable even locally,
ii) stable but not locally parabolic,
iii) parabolic but of local order $\nu < \mu$.

We shall now discuss these three examples in some detail. Let $d = 1, N = 1$.

i) Take for E_h the operator with symbol

$$E_h(\xi) = (1 + i \sin^4 h\xi)(1 - h^2 \sin^2 h\xi) .$$

We have

$$E_h(h^{-1}\xi) = \exp(-h^2\xi^2 + i\xi^4 + o(h^4 + \xi^4)), \quad \xi, h \to 0 , \tag{3.20}$$

so that E_h is consistent with $E(t;P)$ where

$$P(D)u = \frac{\partial^2 u}{\partial x^2} + i \frac{\partial^4 u}{\partial x^4}. \tag{3.21}$$

Then $E(t;P)$ is parabolic of order 2 and

$$\varrho_h(\xi) = (1 + \sin^8 h\xi)^{\frac{1}{2}} (1 - h^2 \sin^2 h\xi) .$$

If for some $\gamma > 0$,

$$\varrho_h(\xi) \leqq 1 + Ck, \qquad |\xi| h \leqq \gamma ,$$

we would have

$$\lim_{h \to 0} \varrho_h(h^{-1}\xi) \leqq 1, \qquad |\xi| \leqq \gamma .$$

This is clearly not the case here for any $\gamma > 0$.

ii) Take now

$$E_h(\xi) = [1 - (h \sin h\xi - \sin^3 h\xi)^2] [1 - \sin^8 h\xi + i \sin^4 h\xi] [1 - \tfrac{1}{16}(1 - \cos h\xi)^4] .$$

We have again (3.20) and so this operator is also consistent with the parabolic operator defined by (3.21). We have for $h \leqq 1$, $0 < |\xi| \leqq \pi$,

$$\begin{aligned} &\varrho_h(h^{-1}\xi) \\ &\quad = [1 - (h \sin\xi - \sin^3 \xi)^2] [1 - \sin^8 \xi (1 - \sin^8 \xi)]^{\frac{1}{2}} [1 - \tfrac{1}{16}(1 - \cos\xi)^4] < 1 , \end{aligned} \tag{3.22}$$

and so the operator E_h is stable. Assume that it were locally parabolic of order ν, so that for some $\gamma > 0$ we have

$$\varrho_h(h^{-1}\xi) \leqq 1 - C_1 h^{4-\nu} |\xi|^\nu + C_2 k, \qquad |\xi| \leqq \gamma . \tag{3.23}$$

Let $\xi_h = \arcsin h^{\frac{1}{2}}$. By (3.23) we would then have

$$\varrho_h(h^{-1}\xi_h) \leqq 1 - C_3 h^{4-\frac{1}{2}\nu}, \qquad h \leqq h_0 ,$$

whereas by (3.22),

$$\varrho_h(h^{-1}\xi_h) = 1 + O(h^4), \qquad h \to 0 .$$

This is a contradiction and so E_h cannot be locally parabolic.

iii) Take this time

$$E_h(\xi) = [1-\sin^4 h\xi\,(h-\sin^2 h\xi)^2 - h^4 \sin^2 h\xi]\cdot$$
$$\cdot[1-\sin^{12} h\xi + i \sin^6 h\xi]\,[1-\tfrac{1}{16}(1-\cos h\xi)^4]\,.$$

We have here

$$E_h(h^{-1}\xi) = \exp(-h^4\xi^2 - h^2\xi^4 + i\xi^6 + o(h^6+\xi^6)), \quad h,\xi \to 0\,,$$

and so E_h is consistent with $E(t;P)$ where

$$P(D)u = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} - i\frac{\partial^6 u}{\partial x^6},$$

so that $E(t;P)$ is parabolic of order 4. We have for $h \leqq h_0$,

$$(3.24) \qquad \varrho_h(h^{-1}\xi)$$
$$= [1-\sin^4\xi\,(h-\sin^2\xi)^2 - h^4\sin^2\xi][1-\sin^{12}\xi\,(1-\sin^{12}\xi)]^{\frac{1}{2}}\,[1-\tfrac{1}{16}(1-\cos\xi)^4]$$
$$\leqq (1-h^4\sin^2\xi)[1-\tfrac{1}{16}(1-\cos\xi)^4]\,,$$

from which it readily follows that E_h is parabolic, and of local order at least 2. However, assume that E_h were parabolic of local order 4. Then we would have for some $\gamma > 0$,

$$(3.25) \qquad \varrho_h(h^{-1}\xi) \leqq 1 - C_1 h^2\xi^4 + C_2 h^6, \qquad |\xi| \leqq \gamma\,.$$

Set as above $\xi_h = \arcsin h^{\frac{1}{2}}$. By (3.25) we would then have

$$\varrho_h(h^{-1}\xi_h) \leqq 1 - C_3 h^4\,,$$

whereas by (3.24),

$$\varrho_h(h^{-1}\xi_h) = 1 + O(h^5), \quad h \to 0\,.$$

This is a contradiction, and so E_h cannot be of local order 4.

The following theorem gives a characterization of parabolic operators E_h which contains at the same time the stability condition and the condition (3.8) on the spectral radius. It is a discrete analogue of Theorem 1.3.

THEOREM 3.4. *The operator E_h is parabolic of order at least ν if and only if there are positive constants C_1, C_2, C_3, h_0, such that for each $h \leqq h_0$, ξ real, there is a positive definite matrix $H_h(\xi)$ with*

$$(3.26) \qquad C_1^{-1} I \leqq H_h(\xi) \leqq C_1 I\,,$$

and

$$(3.27) \qquad |E_h(\xi)|_{H_h(\xi)} \leqq 1 - C_2 k|\xi|^\nu + C_3 k, \qquad h|\xi_j| \leqq \pi\,.$$

PROOF. It is clear in view of Theorem 3.1 that (3.26) and (3.27) imply stability of E_h. Also, (3.27) implies that the condition (3.8) of Theorem

3.2 is satisfied and thus the conditions of the theorem are sufficient for parabolicity of order at least ν.

On the other hand, if E_h is parabolic of order at least ν, then condition (3.8) of Theorem 3.2 holds and by Theorem 3.1 there is a positive definite matrix $H_h(\xi)$ satisfying (3.26) and (3.6). Together, (3.6) and (3.8) give (3.27).

We shall use Theorem 3.4 to prove the following existence theorem:

THEOREM 3.5. *If $E(t;P)$ is parabolic of order μ there exists an operator E_h, which is consistent with $E(t;P)$ and parabolic of order μ.*

PROOF. Set

$$\sin\xi = (\sin\xi_1,\ldots,\sin\xi_d),$$

$$\sigma(\xi) = 1 - d^{-1}2^{-p-1}\sum_{j=1}^{d}(1-\cos\xi_j)^p,$$

and let us consider the explicit operator defined by

$$E_h(\xi) = \sigma(h\xi)I + kP(h^{-1}\sin h\xi). \tag{3.28}$$

Clearly E_h is consistent with $E(t;P)$. We shall prove that if $\lambda = k/h^p$ is small enough, E_h is parabolic of order μ. By Theorem 1.3, since $E(t;P)$ is parabolic of order μ there are positive constants C_1, C_2, C_3, and for each real ξ an hermitean matrix $H(\xi)$ such that

$$C_1^{-1}I \leqq H(\xi) \leqq C_1 I,$$

and

$$2\operatorname{Re}(H(\xi)P(\xi)) \leqq (-2C_2|\xi|^\mu + C_3)H(\xi).$$

We set

$$H_h(\xi) = H(h^{-1}\sin h\xi),$$

and obtain since $\frac{1}{2} \leqq \sigma(\xi) \leqq 1$,

$$\begin{aligned} E_h(\xi)^* H_h(\xi) E_h(\xi) &= \sigma(h\xi)^2 H_h(\xi) + 2k\sigma(h\xi)\operatorname{Re}(H(h^{-1}\sin h\xi)P(h^{-1}\sin h\xi)) + \\ &\qquad + k^2 P^*(h^{-1}\sin h\xi) H_h(\xi) P(h^{-1}\sin h\xi) \\ &\leqq [\sigma(h\xi)^2 - C_2 k|h^{-1}\sin h\xi|^\mu + C_3 k + \\ &\qquad + k^2 C_4(1+|h^{-1}\sin h\xi|^{2p})]H_h(\xi) \\ &\leqq [\sigma(h\xi) + \lambda^2 C_4|\sin h\xi|^{2p} - C_2 k|h^{-1}\sin h\xi|^\mu + C_5 k]H_h(\xi), \end{aligned}$$

and so if we let λ be so small that

$$\sigma(\xi) + \lambda^2 C_4|\sin\xi|^{2p} < 1 \quad \text{for } 0 < |\xi_j| \leqq \pi,$$

we obtain for $h|\xi_j| \leqq \pi$,

$$E_h(\xi)^* \, H_h(\xi) \, E_h(\xi) \leqq (1 - C_6 k |\xi|^\mu + C_7 k) \, H_h(\xi) \, ,$$

which is equivalent to

$$|E_h(\xi)|_{H_h(\xi)} \leqq 1 - C_6 k |\xi|^\mu + C_7 k, \qquad h|\xi_j| \leqq \pi \, .$$

By Theorem 3.4 this proves that E_h is parabolic of order at least μ. By Lemma 3.5 the order of E_h is at most μ. This completes the proof of the theorem. The operator defined by (3.28) was used by Kreiss [8] to prove the existence of stable operators consistent with a correctly posed initial-value problem.

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