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ON THE STRUCTURE OF A CERTAIN CLASS OF LOCALLY COMPACT RINGS

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1. Introduction.

The structure and general properties of compact rings is well known. In comparison, the property of local compactness does not determine the structure of rings, but leaves us with a wide range of possibilities. We therefore often have to do with locally compact rings with additional properties, the most common of which is perhaps boundedness:

For every neighbourhood V of 0 *there exists a neighbourhood U of* 0 *such that, if R is the ring, $RU \subseteqq V$.*

In this note we shall study a class of locally compact rings with still another property. For brevity, we give a name to these rings.

DEFINITION. *A K-ring is a ring satisfying the following conditions*:
a) *local compactness,*
b) *boundedness,*
c) *descending chain condition on ideals containing a given open ideal.*

K-rings are introduced in [7, p. 450] by Kaplansky in order to unify and generalize the classical decomposition of Artin rings [10, IV, § 3, Theorem 3] and his own similar decomposition of compact commutative rings [6, Theorem 17]. For both *compact rings and* (*discrete*) *Artin rings are K-rings.* (The Artin case is trivial, 0 being an open ideal. A compact ring satisfies a) trivially and b) by [6, Lemma 10]. Moreover, in a compact ring the quotient ring modulo any open ideal is discrete and compact, thus finite, and condition c) is obviously satisfied.) Further, Kaplansky shows [7, pp. 450–451] that the decomposition of compact commutative rings (and Artin rings) extends to K-rings.

In this note, we shall study in some more detail the structure of K-rings. It will be proved that a (commutative) K-ring is a direct product (sum) of a compact ring and an Artin ring (Theorem 5). We use the terminology of Nagata [8] and say that a commutative ring with unit and only one

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maximal ideal is *quasi-local.* (A noetherian ring is a ring with unit and a.c.c., and a *local* ring is a noetherian quasi-local ring.) We prove that if a quasi-local K-ring satisfies the 1st axiom of countability it is local (Theorem 6). Finally it is proved that a K-ring with unit is noetherian if and only if it has the radical topology, that is, the topology determined by the powers of the radical (Theorem 9). This theorem also asserts that in a K-ring with unit and open radical, the a.c.c. is equivalent to the 1st axiom of countability.

2. The structure of K-rings.

A radical ring is a ring which coincides with its Jacobson radical. The Jacobson radical of a (commutative) ring is the (set-theoretic) union of all ideals in which every element x has a quasi-inverse, that is, an element y such that $x+y+xy=0$. (It follows that a radical ring has no unit and that the radical in a commutative ring with unit coincides with the intersection of the maximal ideals). Kaplansky's unifying and generalizing theorem mentioned above can now be stated:

THEOREM 1. *A commutative K-ring, R, is (isomorphic and homeomorphic to) the direct product of a radical ring and quasi-local rings.*

The proof of Theorem 1 is sketched in [7, pp. 450–451]. We regard this as sufficient in what concerns the algebraic part. However, we shall prove the topological part of the theorem: that the given ring, and the algebraic direct product are homeomorphic when each factor in the product is taken in its relative topology as a subspace of R and the product is given the product topology. For this we shall need the following well-known fact of which we shall make extensive use throughout the paper:

LEMMA 1. *A locally compact and bounded commutative ring with unit is totally disconnected and has a fundamental system of neighbourhoods of* 0 *consisting of compact open ideals.*

PROOF. The ring is totally disconnected by [7, Theorem 1] as the unit element cannot be a zero-divisor. It then follows from [5, Theorem 3.2] that there is a fundamental system of neighbourhoods of 0 of compact open groups. (The separability condition can be avoided simply by using generalized sequences in lieu of countable ones.) The proof is completed by [6, Lemma 9].

Proof of the topological part of Theorem 1: By the algebraic part of the theorem the ring R is the algebraic direct product (sum) of a radical ring B and a ring with unit C. It follows from the remark on direct sums (products) [7, p. 448] that $R = B + C$ has the product topology. It is then easily seen that both B and C are locally compact (in the relative topology), that they are bounded (since multiplication is direct) and satisfy property c in the definition of K-rings. It is therefore sufficient, by the associativity of the product topology, to prove the topological part for a K-ring with unit. (Indeed, we shall limit ourselves to K-rings with unit throughout the paper, remembering that full generality can always be obtained by adding a radical ring.)

Suppose then, that R is a K-ring with unit. If, by Theorem 1, $\{e_\alpha\}_{\alpha \in I}$ (I an index set) are the units of the quasi-local rings, the algebraic isomorphism g is given by

$$a \in R \to \{ae_\alpha\}_{\alpha \in I} \in \prod_{\alpha \in I} Re_\alpha .$$

We prove that g is a homeomorphism when $\prod_{\alpha \in I} Re_\alpha$ has the proper topology. If Re_α is taken in its relative topology as a subspace of R (or anyone less fine) then

$$\varphi_\alpha\colon \ a \in R \to ae_\alpha \in Re_\alpha$$

is continuous by the continuity of multiplication in R. If therefore $\prod_{\alpha \in I} Re_\alpha$ is given the initial topology relative to the projections π_α (the product topology), we have that $\pi_\alpha \circ g = \varphi_\alpha$ is continuous for all α and g is continuous by [1, § 2, Prop. 4]. It remains to be proved that g is open. (If R were compact this would follow from the Tychonoff theorem and the fact that a map between compact Haussdorff spaces is a homeomorphism). By Lemma 1 it is sufficient to prove that the set

$$g(A) = \prod_{\alpha \in I} Ae_\alpha = \prod_{\alpha \in I} A \cap Re_\alpha$$

is open for all open ideals A. Since by the definition of a K-ring R/A has d.c.c. (property c), $A \cap Re_\alpha$ must equal Re_α but for a finite number of α's. But then $g(A)$ is open in the product topology when each factor is taken in the topology induced from R(or finer).

In the remainder of the paper we shall study in greater detail the structure and properties of K-rings. We drop the word "commutative" in later connections, always assuming that the rings considered are commutative. First we consider the role of property c. We have seen that it is essential to the homeomorphism in Theorem 1. It also fits together with the properties of products of locally compact spaces to give

THEOREM 2. *The algebraic direct product of K-rings with unit, whereof all but a finite number are compact, is itself a K-ring* (*with unit*) *in the product topology.*

PROOF. The product is locally compact by [1, § 9, Prop. 14]. By Lemma 1, each factor has a topology defined by ideals. A fundamental system of neighbourhoods of 0 in the product then consists of products of finite numbers of (factor) ideals and full rings. As these products are ideals in the product ring this ring is bounded. As the factors of these products are full rings except a finite number, we see that the product ring has property *c*.

The converse of this theorem is also true. The property of being a K-ring is inherited by the factors of the decomposition given in Theorem 1.

THEOREM 3. *A K-ring with unit, R, is the direct product of quasi-local K-rings. All of these, but a finite number, are compact.*

PROOF. We know by Theorem 1 that R is the direct product, in the topological sense, of ideals in R with the topology induced from R. We know [1, § 9, Prop. 14] that the factors are locally compact in this topology and that all but a finite number are compact. By Lemma 1 R has a fundamental system of neighbourhoods of 0 consisting of ideals. This property is obviously inherited by any ideal in R in its relative topology, in particular by each factor, and it easily follows that the factors are bounded in their relative topology.

Now, suppose I is a factor ideal in R, and let $\{A_n\}_{n\geqq 1}$ be a descending chain of ideals in I (they are also ideals in R since I is a factor in R) which all contain the open ideal (in I's relative topology) $A = B \cap I$ where B is an ideal open in R. Then $\{B + A_n\}_{n \geqq 1}$ is a chain of ideals in R which breaks off by property *c*, say at $B + A_N$. It follows by the modular law that for all $n \geqq N$ we have

$$\begin{aligned} A_n = A_n + (B \cap I) &= I \cap (A_n + B) \\ &= I \cap (A_N + B) = A_N + (I \cap B) = A_N . \end{aligned}$$

Thus I has property *c*.

As mentioned in the introduction, the class of K-rings includes all compact and Artin rings. And we easily find a large class of K-rings which are neither compact nor Artin, namely any direct sum (product) of a (non-Artin) compact ring and an (infinite) Artin ring. It turns out that this exhausts the possibilities.

THEOREM 4. *A non-discrete, quasi-local K-ring is compact.*

PROOF. Let R be the ring. By Lemma 1, R has a fundamental system of neighbourhoods of 0 of compact ideals. If R is non-discrete, there exists a compact ideal A and an open (and compact) ideal $B \neq A$ such that $B \subseteq A$. The residue classes of A mod B is an open covering of A by disjunct sets, thus finite in number since A is compact. That is, A/B is finite. We show that the field R/M, where M is the unique maximal ideal in R, has not more elements than A/B which proves that it is finite. Suppose $a, b \in R$ and that $a \not\equiv b \bmod M$. For any $x \in A \cap \complement B$ we then have $ax \not\equiv bx \bmod B$. For if not, that is, if $(a-b)x \in B$, we would have

$$x = (a-b)^{-1}(a-b)x \in B ,$$

since $a-b \notin M$ and is a unit and B is an ideal. This is impossible.

The residue class ring $\bar{R} = R/A$ has d.c.c. since R has the property c. The maximal ideal in $\bar{R}$ is $\bar{M} = M/A$ and $\bar{R}/\bar{M} = R/A\big/M/A \cong R/M$ and therefore finite. $\bar{R}$ is noetherian, thus

$$\bigcap_{s \geqq 1} \bar{M}^s = \{\bar{0}\}$$

by [10, IV, § 7, Theorem 12], and as $\bar{R}$ has d.c.c. it follows that there exists an integer s such that $\bar{M}^s = \{\bar{0}\}$. Using Lemma 4 from [9] $s-1$ times we derive that $\bar{R}/\bar{M}^s = \bar{R}$ is finite. This means that R is a finite union of compact sets and therefore compact.

THEOREM 5. *A ring with unit is a K-ring if and only if it is the direct product of compact rings and a finite number of local Artin rings.*

PROOF. The "if" part is seen directly or from Theorem 2. The "only if" part follows immediately from Theorems 3 and 4.

COROLLARY. *Any K-ring with unit is the direct sum (product) of an Artin ring and a compact ring.*

3. 1st axiom of countability.

In [6] Kaplansky gives an example of a compact quasi-local, non-local ring. The possibilities of constructing such rings are limited by [6, Theorem 20] which states that a quasi-local compact ring in which the second power of the radical is open, is local and has the radical topology. Another condition with "localizing" effect is the 1st axiom of countability.

LEMMA 2. *In a K-ring with unit the radical topology is equal to or finer than the topology of the ring.*

PROOF. If R is the ring and N the radical (the intersection of the maximal ideals) we have by the same reasoning with respect to N as in the proof of Theorem 4 with respect to M that, if A is an open ideal, then $\bar{N}^s = \{\bar{0}\}$ for suitable s, that is, $N^s \subseteq A$. It then follows from Lemma 1 that every open neighbourhood of 0 contains a power of the radical.

THEOREM 6. *A quasi-local K-ring R which satisfies* 1*st axiom of countability is local.*

PRROF. If R is discrete, there is nothing to prove. If not, it is compact by Theorem 4. Using Lemma 1 we see that its maximal ideal M is open, hence closed and compact. Since $M \times M$ is compact in $R \times R$ it follows by the continuity of multiplication that M^2 is compact and by induction that M^s is compact for all s. Since R is complete, this and Lemma 2 show that the premises of [1, Ch. II, § 3, Prop. 7] are satisfied and it follows that R is complete in its radical topology. It then follows from [8, Theorem 31.1] that, if I is an index set of the same cardinality as a set of generators for the maximal ideal, then R is isomorphic to a ring of formal power series in the entities $\{X_\lambda\}_{\lambda \in I}$ with coefficients from a complete local ring.

We show that I is a finite set. The theorem then follows from [4, Theorem 3].

Let $\{A_n\}_{n \geqq 1}$ be a decending chain of ideals which form a fundamental system of neighbourhoods of zero. R/A_n is finite since R is compact. Furthermore, we have for all n, that if $X_\lambda, X_\mu \notin A_n$ and $X_\lambda \neq X_\mu$, then $X_\lambda \not\equiv X_\mu \mod A_n$. (There are no relations between the X_λ's.) This implies that for all n the number of X_λ's in $\complement A_n$ must be finite. Accordingly, the series $\{c_n\}_{n \geqq 1}$ where

$$c_n = \sum_{X_\lambda \notin A_n} X_\lambda$$

is well defined and we easily see that it is a Cauchy-series which converges towards the sum of all the entities X_λ. This sum, however, is homogeneous of 1st degree and is a formal power series only if it is finite. Thus I must be finite.

4. Noetherian K-rings.

Theorem 20 [6] entails that if a compact quasi-local K-ring has the radical topology it is local. We shall prove a kind of converse to this, namely that a noetherian K-ring has natural topology. In doing this we shall follow a line of proof which is independent of Theorem 1 and its analogue for compact rings [6, Theorem 17]. This will also lead to a determination of the structure of noetherian K-rings based on Lemma 1 alone. (Using Theorem 1 and its consequences, we can prove the results in this section rather easily.) Finally, we shall make clear the interrelations between a.c.c., radical topology and 1st axiom of countability in a K-ring with unit.

We start with showing the significance of property c in a noetherian ring.

LEMMA 3. *A noetherian topological ring R has the property c if and only if every open prime ideal is maximal.*

PROOF. Suppose every open prime ideal in R is maximal and let A be any open ideal. Every prime ideal containing A is open (a group with an inner point is open), thus maximal. This means that every prime ideal in R/A is maximal and R/A has d.c.c. by [10, IV, § 2, Theorem 2]. Conversely, suppose R has property c and let P be any open prime ideal. For some maximal ideal M, we have $P \subseteq M$ and by [10, IV, § 7, Theorem 12′],

$$\bigcap_{s \geqq 1} P + M^s = P .$$

By property c then, $P + M^s \subseteq P$, $M^s \subseteq P$ for some s. Since P is prime and M maximal, we have that $P = M$.

LEMMA 4. *If A and B are ideals in a ring R with unit, if A is compact and B finitely generated, then the ideal AB is compact.*

PROOF. Say $b_1, b_2, \ldots, b_n$ generate B. Then

$$AB = A(Rb_1 + Rb_2 + \ldots + Rb_n) = Ab_1 + Ab_2 + \ldots + Ab_n .$$

which is compact being the continuous image of a compact set.

LEMMA 5. *In a locally compact and bounded noetherian ring R, the radical is open.*

PROOF. The theorem is proved if we can prove the existence of a compact open ideal A such that $\bigcap_{s \geqq 1} A^s = \{0\}$. It then follows that the filter base $\{A^s\}_{s \geqq 1}$ has 0 as its unique point of adherence (the ideals A^s are all closed by Lemma 4 and induction) and therefore converges to 0 by [1, § 9, Corollary of Theorem 1]. Being a locally compact group, R is complete [2, § 3, Prop. 4] and we deduce from [6, Theorem 12] that the open ideal A is contained in the radical, hence the radical is open.

Assume therefore, that $\bigcap_{s \geqq 1} A^s \neq \{0\}$ for all compact open ideals A. This means, [10, IV: § 7, Theorem 12 and § 6, Theorem 11, Corollary 3] that

$$(1 + A) \cap \bigcup_{P \in \mathfrak{P}} P \neq \emptyset$$

for all compact open ideals A when $\mathfrak{P}$ is the set of associated prime ideals of 0. Using Lemma 1 we derive that

$$1 \in \overline{\bigcup_{P \in \mathfrak{P}} P} = \bigcup_{P \in \mathfrak{P}} \bar{P}$$

(0 has a finite number of associated prime ideals) and thus $1 \in \bar{P}$ for some P. Now, if B is some compact open ideal we have

$$B = B\bar{P} \subseteq \overline{BP} = BP$$

by the continuity of multiplication and Lemma 4. But $B \subseteq BP \subseteq P$ implies, since B is open, that P is open and closed. Hence $P = \bar{P} = R$ which is impossible.

THOEREM 7. *A noetherian K-ring is semi-local and the topology is the radical topology.*

PROOF. Let R be the ring and N its radical. By Lemma 5 N is open and it follows by Lemma 3 that N's associated prime ideals $P_1, P_2, \ldots, P_n$ are maximal. As R is noetherian N contains some power of its radical ($N^{\frac{1}{2}}$), say

$$(P_1 \cap P_2 \ldots \cap P_n)^s = (P_1 \cdot P_2 \cdot \ldots \cdot P_n)^s \subseteq N .$$

For any maximal ideal M then, it follows since M is prime and since N is contained in every maximal ideal that $P_i \subseteq M$ for some i, $i = 1, 2, \ldots, n$. Then, since P_i is maximal, $M = P_i$. Hence all the maximal ideals in R are among the P_i, $i = 1, 2, \ldots, n$, and R is semi-local.

It remains to be shown that the topology in R is the radical topology. By Lemma 2 it is sufficient to prove that R's topology is finer than the radical one. Let A be a compact open ideal contained in N with $N^{s_0} \subseteq A$ (Lemma 2) and let B be the closure of N^{s_0}. For all s, we have

$$N^{s_0 s} \subseteq B^s \subseteq N^s .$$

Thus the family $\{B^s\}_{s \geqq 1}$ form a fundamental system of neighbourhoods of 0 for the radical topology of R. Hence $B \subseteq A$ is compact, and it follows by Lemma 4 and induction that B^s is closed for all s. The ring R/B^s, therefore, is Hausdorff in its quotient topology for all s. We shall prove that for all s, R/B^s is discrete, thus B^s open which completes the proof.

For all s, since $B^s \supseteq N^{s_0 s}$ the associated prime ideals of B^s are the maximal ideals in R and R/B^s has d.c.c. by [10, IV, § 2, Theorem 2]. In its quotient topology, R/B^s has a fundamental system of neighbourhoods of 0 consiting of ideals, viz. the images of the open ideals in R containing B^s. Being Hausdorff, it follows by considering a maximal descending chain of open ideals that it is discrete.

As locally compact groups are complete [2, § 3, Prop. 4] we can apply a theorem of Chevalley [3, Prop. 2] to get the structure of noetherian K-rings.

THEOREM 8. *A noetherian K-ring R is the finite direct product of local rings.*

PROOF. It follows from Theorem 7 and the theorem of Chevalley that R has orthogonal idempotents e_i, $i=1,2,\ldots,n$, such that $R=Re_1+Re_2+\ldots+Re_n$.

The 1st axiom of countability and the condition that a K-ring has its radical topology can now be related to the property of being noetherian:

THEOREM 9. *In a K-ring, R, with unit the following conditions are equivalent.*

1) *The ring has* a.c.c.
2) *The topology in the ring is the radical topology.*
3) *The topology in the ring satisfies* 1*st axiom of countability and the radical is open.*

PROOF. We prove that 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 1). As 1) $\Rightarrow$ 2) is a part of Theorem 7 and 2) $\Rightarrow$ 3) is trivial since neighbourhoods are translated from 0, we only consider the implication 3) $\Rightarrow$ 1).

By Theorem 3, a K-ring is the direct product of quasi-local K-rings and the topology is the product topology. If this topology satisfies 1st axiom of countability each factor in the product must satisfy the countability condition and is noetherian by Theorem 6. Further the product must be finte, for if not, the descending chain of ideals $\{A_n\}_{n\geq 1}$, where

$$A_n = \bigcap_{i=1}^{n} M_i, \quad M_i \text{ maximal ideal in } R\ ,$$

would not break off in contradiction with property c (in the definition of a K-ring) and the fact that the radical is contained in A_n for all n. Being a finite product of noetherian rings, R is noetherian.

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