EXPANSIVE AUTOMORPHISMS IN COMPACT GROUPS

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An automorphism \( \theta \) of a topological group \( G \) is expansive iff there is neighborhood \( V \) of the identity such that for any two distinct elements \( x, y \in G \), there is an integer \( n \) with \( \theta^n(xy^{-1}) \notin V \). For the notions of expansive homeomorphism on topological space, see [1], [2].

In [1], M. Eisenberg proved that when \( G \) is a compact connected Lie group and \( G \) admits an expansive automorphism, then \( G \) is abelian. He also shows the existence of expansive automorphism on \( n \)-dimensional torus group for all positive integer \( n > 1 \). In this note, we shall prove the following theorem.

**Theorem.** If \( F \) is a compact connected finite dimensional topological group and \( G \) admits an expansive automorphisms, then \( G \) is abelian.

**Proof.** Let \( G \) be a compact connected finite dimensional topological group. It is known that \( G \) is isomorphic to \( (L \times H)/D \), where \( L \) is a compact simply connected semi-simple Lie group, \( H \) is compact connected abelian group, and \( D \) is a finite normal subgroup of the direct product \( L \times H \). (Cf. [4, Example 107]). Since \( D \) is finite,

\[
\frac{LD}{D} \approx \frac{L}{L \cap D}
\]

is a compact connected semi-simple Lie group. (Accurately, we should write \( (L \times \{ \mu \})D \) for \( LD \), where \( \mu \) is the identity of \( H \).) Let \( \theta \) be an automorphism of \( (L \times H)/D \). Then \( (LD)/D \) is invariant under \( \theta \). This can be seen by the following diagram:

\[
\begin{array}{ccc}
L \times H & \xrightarrow{\pi} & H \\
\varphi \downarrow & & \downarrow \psi \\
(L \times H)/D & \xrightarrow{F} & H/\pi(D)
\end{array}
\]

Here, \( \pi \) is the projection, \( \varphi \) and \( \psi \) are the quotient maps, \( F \) is defined by

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Received August 20, 1965.
This research was partially supported by NASA of U.S.A., NGR 10-007-005.
\[ F(\varphi(l,h)) = \varphi(h). \] One could show that \( F \) is well defined and \( F \) is a continuous homomorphism. Since \( \theta((LD)/D) \) is also a compact connected semi-simple Lie group and \( H/\pi(D) \) is abelian, it follows that \[ F \theta((LD)/D) = \pi(D) \in H/\pi(D). \] Since \( \ker F = (LD)/D \) we get

\[ \theta((LD)/D) \subseteq (LD)/D. \]

Hence \( \theta \) induces an automorphism on \( (LD)/D \). Now, by a theorem in [3], the bi-continuous automorphism group \( A((LD)/D) \) of \( (LD)/D \) is compact when \( A((LD)/D) \) is topological by compact open topology. So the natural action of \( A((LD)/D) \) on \( (LD)/D \) is equicontinuous, a fortiori, no automorphism on \( (LD)/D \) can be expansive unless \( (LD)/D \) is degenerate. Thus, we can conclude if \( \theta \) is an expansive automorphism on \( G \), then \( (LD)/D \) is degenerate, and \( G \) is abelian. The proof is complete.

REFERENCES


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