INFINITE KUMMER EXTENSIONS

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Let $n$ be a positive integer and let $F$ be a field which contains $n$ distinct $n^{th}$ roots of unity. Denote by $F^*$ the multiplicative group of non-zero elements of $F$ and denote by $(F^*)^n$ the subgroup of $n^{th}$ powers of elements in $F^*$. Suppose $C$ is an algebraic closure of $F$. The classical theory of Kummer extensions establishes a 1-1 correspondence between the subfields $K$ of $C$ which are finite abelian extensions of $F$ of exponent dividing $n$ and the subgroups $Q$ of $F^*$ containing $(F^*)^n$ and such that $Q/(F^*)^n$ is finite. The correspondence is such that if $K$ corresponds to $Q$ then $G(K/F)$, the Galois group of $K/F$, is isomorphic to $Q/(F^*)^n$. For an exposition of these results see [1].

The object of this paper is to extend these results to arbitrary (not necessarily finite) abelian extensions $K$ of $F$ whose Galois group is of bounded order $n$, i.e. if $\sigma \in G(K/F)$ then $\sigma^n = 1$.

Thus let $K$ be a subfield of $C$ containing $F$ such that $K/F$ is Galois and $G = G(K/F)$ is abelian of bounded order $n$. Define

$$S = S(K) = \{ \alpha \in K^* : \alpha^n \in F^* \}.$$ 

For $\alpha \in S$ define a function $\chi_\alpha$ on $G$ with values in $K^*$ by the rule

$$\chi_\alpha(\sigma) = \alpha/\sigma(\alpha).$$

Just as in the ordinary theory of Kummer extensions it is easy to check that $\chi_\alpha(\sigma) \in Z$ and $\chi_\alpha$ is a homomorphism of $G$ into $Z$ where $Z$ denotes the group of $n^{th}$ roots of unity in $F^*$. Further, the map $\alpha \rightarrow \chi_\alpha$ is a homomorphism of $S$ into $X$, the group of characters on $G$. Note that since $G$ is of bounded order any character on $G$ is automatically continuous in the Galois topology on $G$.

Now let $\chi$ be any character on $G$. Since $G$ is of bounded order $n$, one may assume that $\chi$ takes its values in $Z$. The map $\sigma \rightarrow \chi(\sigma)$ is continuous and satisfies

$$\sigma \tau \rightarrow \chi(\sigma) \chi(\tau) = \chi(\sigma) \sigma(\chi(\tau)).$$

Thus $\sigma \rightarrow \chi(\sigma)$ is a continuous cocycle for the system $(G, K^*)$. Since $H^1(G; K^*) = 1$ (see [2]) there exists $\beta \in K^*$ such that

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\[ \chi(\sigma) = \beta / \sigma(\beta). \]

Since \( \chi(\sigma)^n = 1 \) for all \( \sigma \in G \) one has \( \sigma(\beta^n) = \beta^n \) for all \( \sigma \in G \). Thus \( \beta \in S \).

This shows that the map \( \alpha \to \chi_\alpha \) of \( S \) into \( X \) is surjective. The kernel of this map is \( F^* \) and therefore \( S/F^* \approx X \).

Now set
\[ Q = Q(K) = \{ \alpha^n : \alpha \in S \}. \]

Then \( Q/(F^*)^n \) is isomorphic to \( S/F^* \). Thus given an abelian extension \( K \) of \( F \) of bounded order \( n \) one can associate the group \( Q/(F^*)^n \). According to the Pontriagin duality theorem then \( G \) is topologically isomorphic to the character group of \( Q/(F^*)^n \).

Note that \( Q \) also has the property that \( K \) is generated over \( F \) by
\[ S = Q^{1/n} = \{ \alpha \in K^* : \alpha^n \in Q \}. \]

For let \( K' = F(S) \) and let \( \sigma \in G(K/K') \). Then for any \( \chi_\alpha \in X \),
\[ \chi_\alpha(\sigma) = \alpha/\sigma(\alpha) = 1. \]

The duality theorem gives that \( \sigma = 1 \). Therefore \( K' = K \).

It remains to show that any subgroup \( Q \) of \( F^* \) containing \((F^*)^n \) is a \( Q(K) \) for some field \( K \) and that the correspondence \( K \to Q(K) \) is 1-1. Given such a \( Q \) set
\[ Q^{1/n} = \{ \alpha \in C : \alpha^n \in Q \}. \]

and take \( K = F(Q^{1/n}) \). Then \( K \) is the splitting field over \( F \) of the set of polynomials \( \{ X^n - \beta : \beta \in Q \} \). Since each of these polynomials is separable \( K \) is a normal separable extension of \( F \). On the other hand \( K = \lim \uparrow E \) where \( E \) is the splitting field over \( F \) of some finite set of polynomials
\[ \{ X^n - \beta_i : \beta_i \in Q, i = 1, 2, \ldots, s \}. \]

Therefore \( G(K/F) = \lim \downarrow G(E/F) \). By the theory of ordinary Kummer extensions each \( G(E/F) \) is finite abelian of exponent dividing \( n \). Therefore \( G(K/F) \) is abelian of bounded order \( n \). Moreover, the field \( K \) just constructed has the property that
\[ Q(K) = \{ \alpha^n : \alpha \in S(K) \} = Q; \]

for let \( R \) be a subgroup of \( F^* \) such that
\[ Q \supset R \supset (F^*)^n \]

and \( R/(F^*)^n \) is finite. Then \( Q = \lim \uparrow R \). If \( E \) is the subfield of \( C \) obtained by adjoining the set \( \{ \alpha \in C : \alpha^n \in R \} \) to \( F \), then the theory of finite Kummer extensions gives that \( Q(E) = R \). Therefore \( Q(K) = \lim \uparrow Q(E) = \lim \uparrow R = Q \).
Finally, since any abelian extension $K$ of bounded order $n$ can be described by $K = F(S(K))$ it follows that the correspondence $K \rightarrow Q$ is 1-1. These considerations give the following

**Theorem.** There is a 1-1 correspondence between the abelian extensions $K$ of $F$ of bounded order $n$ and the subgroups $Q$ of $F^*$ containing $(F^*)^n$. This correspondence has the property that if $K$ corresponds to $Q$ then $G(K/F)$ is topologically isomorphic to the character group of $Q/(F^*)^n$.

**References**


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