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EXTREMELY AMENABLE SEMIGROUPS

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1. Introduction.

Let S be a discrete semigroup, $m(S)$ the space of bounded real functions on S with the usual sup norm, and $m(S)^*$ be the conjugate Banach space of $m(S)$. An element $\varphi \in m(S)^*$ is a mean if $\varphi(f) \geqq 0$ whenever $f \geqq 0$ and $\varphi(1)=1$ where 1 is the constant one function on S. The semigroup S is said to be left amenable, denoted by LA, if there is a mean φ in $m(S)^*$ which is in addition left invariant i.e. satisfies $\varphi(f_a)=\varphi(f)$ for any f in $m(S)$ and a in S (where $(f_a)(s)=f(as)$ for any s in S). We say in this case that S, or $m(S)$, admits a left invariant mean. S is said to be *extremely left amenable*, denoted by ELA, if $m(S)$ admits a left invariant mean φ which is in addition multiplicative, that is $\varphi(fg)=\varphi(f)\varphi(g)$ for any f,g in $m(S)$.

The first to consider extremely amenable semigroups was T. Mitchell[1] in [14] who proved among others the following interesting result: The semigroup S admits a multiplicative left invariant mean if and only if it has the common fixed point property on compacta (i.e. for each compact Hausdorff space X and for each homomorphic representation S' of S as a semigroup (under functional composition) of continuous maps of X into itself, there is some x_0 in X such that $s'(x_0)=x_0$ for all s' in S'). It has been shown by Mitchell [14] that if S has the property that each two elements of S have a common right zero (i.e. if a, $b \in S$ then $ac=bc=c$ for some c in S) then S admits a multiplicative left invariant mean. The converse statement has been shown by Mitchell [14] to hold true only under the additional hypothesis that S has left cancellation or is abelian. Mitchell even conjectures in the introduction of [14] that this converse implication does not hold true in general, i.e. that there exists an ELA semigroup which does not have the property that each two of its elements have a common right zero.

We get as a consequence of the theorems proved in this work, that Mitchell's conjecture is not correct, i.e. the property that each two ele-

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[1] Many thanks are due to T. Mitchell for kindly letting us have preprints of his works.

ments of S have a common right zero characterizes the semigroups which admit multiplicative left invariant means.

If $a \in S$, denote by 1_a that element of $m(S)^*$ which satisfies $1_a f = f(a)$ for each f in $m(S)$. Elements of $\{1_a;\ a \in S\}$ are said to be point measures. If $a \in S$ and $\varphi \in m(S)^*$, let $L_a \varphi \in m(S)^*$ be such that $(L_a \varphi) f = \varphi f_a$ for any f in $m(S)$. We write sometimes $l_a f$ instead of f_a.

We have furthermore the following Day–Følner type characterization of extremely left amenable semigroups (see Day [4, pp. 524–525] and Følner [9]). (For a beautiful proof of Følner's theorem see Namioka [16].)

THEOREM. *The semigroup S admits a multiplicative left invariant mean if and only if there is a net of point measures $\{1_{s_\alpha}\}$ which converges in norm to left invariance* i.e. *such that*

$$\lim_\alpha \|L_a 1_{s_\alpha} - 1_{s_\alpha}\| = 0 \quad \textit{for each } a \in S\,.$$

(The last condition implies immediately that any two elements of S have a common right zero.)

If S is a discrete semigroup then the convolution multiplication $\odot$ renders $\beta(S)$ (the Stone–Čech compactification of S or the set of multiplicative means on $m(S)$) a semigroup. Then $\beta(S)$ has a right zero if and only if S is ELA. Combining part of Mitchell's results in [14] together with those obtained in this paper one has the

THEOREM (*characterization*). *The following conditions on a semigroup S are equivalent*:

1. *S has the common fixed point property on compacta.*
2. *S admits a multiplicative left invariant mean.*
3. *$\beta(S)$ has a right zero.*
4. *There is a net $\{1_{s_\alpha}\}$ of point measures which converges in norm to left invariance.*
5. *Each two elements of S have a common right zero.*

Condition 5 and hence each one of the above conditions implies (and is not in general implied by) that each element of S has a right zero.

As pointed out in Mitchell [14], it is shown in Ljapin [13, p. 66] that a semigroup S has the property that any element has a right zero if and only if it has the individual fixed point property on any set, i.e. in an arbitrary representation of the semigroup by transformations on *any set* (under functional composition) every transformation has a fixed point (which may depend on the transformation). The last part of the theorem shows that the common fixed point property on compacta for S implies that S has the individual fixed point property on any set. It is in fact this implication which was conjectured to be false by Mitchell.

If $A \subset S$ then $\varphi \in m(S)^*$ is said to be A-left invariant if $L_a\varphi = \varphi$ for each a in A. If A consists of one element b, say, then we say that φ is b-left invariant. If $B \subset S$ and $\varphi \in m(S)^*$ we write sometimes $\varphi(B)$ instead of $\varphi(1_B)$ where 1_B is the function on S which is one on B and zero otherwise. 1 will stand for 1_S sometimes.

If S is a left or right cancellation semigroup, then $c \in S$ has order $n \geqq 1$ if $n+1$ is the first integer which satisfies $c^{n+1} = c$. An element c has infinite order if no such $n \geqq 1$ exists.

The following two theorems which are interesting for their own sake, we think, are in fact the crux of the proof of the above theorems:

THEOREM. *Let S be a right cancellation semigroup and $c \in S$ be an element of infinite order. If $\varphi \in m(S)^*$ is a c-left invariant mean then $\varphi(A)$ takes at least all rational values in the closed unit interval when A ranges over all subsets of S, which are left c-almost convergent*[2].

THEOREM. *Let S be a right cancellation semigroup and $c \in S$ be an element of order $n \geqq 2$. If $\varphi \in m(S)^*$ is a c-left invariant mean then $\varphi(A)$ takes at least all values $r = k/n$, $k = 0, 1, \ldots, n$ when A ranges over all subsets of S, which are left c-almost cenvergent.*

CONSEQUENCE. *Let S be a right cancellation* ELA *semigroup. Then S is the trivial group containing identity only. (If $\varphi \in \beta(S)$ then $\varphi(A) = 0$ or 1 only.) In particular among groups, only the trivial group is* ELA.

This consequence has been proved by T. Mitchell in [14] for the case that S has two sided cancellation. As pointed out in Mitchell [14] (after proof of theorem 3) his method runs into difficulty if two sided cancellation is replaced by right cancellation. Using the preceding two theorems we give a simple proof for the characterization theorem. We, furthermore, investigate the class of ELA semigroups in analogy to M. Day's investigation of amenable semigroups in [4].

The fact that the property ELA for a semigroup S is equivalent to the property that any two elements of S have a common right zero enables us to give very simple proofs to the following results which have analogues for amenable semigroups (see Day [4, pp. 515–517] and [10, p. 50]):

(a) If S is ELA, so is any homomorphic image (due to Mitchell [14]).

[2] $A \subset S$ is left c-almost convergent if $\varphi_1(A) = \varphi_2(A)$ for any two c-left invariant means φ_1, φ_2.

(b) If S is a semigroup with $S = \bigcup_{t \in T} S_t$ where S_t are ELA semigroups and for any $t_1, t_2 \in T$,

$$S_{t_1} \cup S_{t_2} \subset S_{t_3} \quad \text{for some} \quad t_3 \in T\ ,$$

then S is ELA.

(c) If S is an ELA semigroup then any countable subsemigroup is included in a countable ELA subsemigroup.

The analogy between ELA semigroups and amenable semigroups breaks down when we come to full direct products. While as shown by Day [4, p. 517] the full direct product of even left amenable groups need not be left amenable we have here that the full direct product of ELA semigroups *is* ELA.

We ask now the following question: Let S be a ELA semigroup. Is each finitely generated subsemigroup contained in a *finitely* generated ELA subsemigroup? We answer this question in the negative by giving an explicit example of a semigroup S with the following surprising properties:

S is a countable ELA semigroup such that any subsemigroup which can be included in a finitely generated subsemigroup (and a fortiori any finitely generated subsemigroup) is *not* ELA while the remaining subsemigroups, i.e. those which cannot be included in finitely generated subsemigroups, *are* ELA. Surprisingly enough *each* subsemigroup of S is left amenable and even moreover admits an infinite dimensional set of left invariant means (none of which is multiplicative, if the subsemigroup can be included in a finitely generated subsemigroup). Furthermore S has left cancellation, is not right amenable, and does not contain elements of finite order.

In view of this example it is interesting to note that such a behaviour is impossible if S contains "enough" periodic elements ($c \in S$ is periodic if $c^{2n} = c^n$ for some $n \geqq 1$). In fact we have:

PROPOSITION. *Let S be an* ELA *semigroup such that each of its right ideals contains a periodic element. If S_0 is a finitely generated subsemigroup, generated by $\{s_1, \ldots, s_n\}$ say, then there is some s_{n+1} in S such that $\{s_1, \ldots, s_n, s_{n+1}\}$ generates a* ELA *subsemigroup.*

2. The characterization theorem.

We begin with the following known facts:

If S is a semigroup, $c \in S$ and $\varphi \in m(S)^*$ is any c-left invariant mean then $\varphi(cA) \geqq \varphi(A)$ for any $A \subset S$, since $(l_c 1_{cA})(s) = 1_{cA}(cs) \geqq 1_A(s)$ for any

s in S. In particular if φ is a left invariant mean on S and B any right ideal then $bS \subset B$ if $b \in B$ which implies that $\varphi(B) = 1$. Therefore, if S is LA, any two right ideals have nonvoid intersection.

LEMMA 1. *Let S be a right cancellation semigroup, $c \in S$ an element of infinite order and $\emptyset \neq V \subset S$ be such that $cV \subset V$. Then for any positive integer k there exist subsets $A_1, \ldots, A_k$ such that:*

(i) $\bigcup_1^k A_i = V$.
(ii) $A_i \cap A_j = \emptyset$ if $i \neq j$.
(iii) $cA_i \subset A_{i+1}$ if $i \leqq k-1$ and $cA_k \subset A_1$.

PROOF. If $c^i = c^j$ for $i \neq j$ then $c^m = c$ for some $m > 1$ which cannot be. Hence $\{c^n\}_1^\infty$ consists of different elements. Let $a \in V$ and consider the sets $V_i(a)$ defined by

$$\begin{aligned} V_1(a) &= \{ca, c^{k+1}a, c^{2k+1}a, \ldots, c^{jk+1}a, \ldots\}, \\ V_2(a) &= \{c^2a, c^{k+2}a, \ldots, c^{jk+2}a, \ldots\}, \\ &\vdots \\ V_k(a) &= \{c^k a, c^{2k}a, \ldots, c^{jk+k}a, \ldots\}. \end{aligned}$$

Then $V_i(a) \cap V_j(a) = \emptyset$ if $i \neq j$,

$$\bigcup_1^k V_i(a) \subset V, \quad cV_i(a) \subset V_{i+1}(a), \quad cV_k(a) \subset V_1(a).$$

Let now $\mathscr{V}$ be the set of all ordered k-tuples $(U_1, \ldots, U_k)$ where $\emptyset \neq U_i$ are subsets of S,

$$\bigcup_1^k U_i \subset V, \quad U_i \cap U_j = \emptyset \text{ if } i \neq j,$$
$$cU_i \subset U_{i+1} \text{ if } i \leqq k-1, \quad cU_k \subset U_1.$$

We have shown that $\mathscr{V} \neq \emptyset$. We partially order $\mathscr{V}$ by

$$(V_1, \ldots, V_k) \geq (U_1, \ldots, U_k) \quad \text{iff} \quad U_i \subset V_i, \ 1 \leqq i \leqq k.$$

Let $\{(U_1^\alpha, \ldots, U_k^\alpha);\ \alpha \in I\}$ be a linearly ordered subset of $\mathscr{V}$ and let $U_i^0 = \bigcup_\alpha U_i^\alpha$ for $1 \leqq i \leqq k$. It is clear that $cU_i^0 \subset U_{i+1}^0$ for $1 \leqq i \leqq k-1$ and $cU_k^0 \subset U_1^0$. Moreover $U_i^0 \cap U_j^0 = \emptyset$ if $i \neq j$ since

$$U_i^0 \cap U_j^0 \neq \emptyset \quad \text{implies} \quad U_i^\alpha \cap U_j^\beta \neq \emptyset$$

for some α, β in I. Now either $(U_1^\alpha, \ldots, U_k^\alpha) \geq (U_1^\beta, \ldots, U_k^\beta)$ or the reversed inequality holds. In the first case we would have

$$\emptyset \neq U_i^\alpha \cap U_j^\beta \subset U_i^\alpha \cap U_j^\alpha$$

which cannot be. The reversed inequality would imply that $U_i^\beta \cap U_j^\beta \neq \emptyset$ which again cannot be. Hence $(U_1^0,\ldots,U_k^0)\in\mathscr{V}$ and obviously $(U_1^0,\ldots,U_k^0)\geq(U_1^\alpha,\ldots,U_k^\alpha)$ for each $\alpha\in I$. Since any linearly ordered subset of $\mathscr{V}$ has an upper bound we get by Zorn's lemma that $\mathscr{V}$ contains a maximal element say $(A_1,\ldots,A_k)$. We show now that $V=\bigcup_1^k A_i$. In fact assume that $a\in V$ and $a\notin\bigcup_1^k A_i$. Then two cases may occur:

1. $\{c^n a\}_1^\infty\cap\bigcup_1^k A_i=\emptyset$. In this case define $A_i^*=A_i\cup V_i(a)$ for $1\leqq i\leqq k$ Then $A_i^*\cap A_j^*=\emptyset$ if $i\neq j$ since

$$V_i(a)\cap V_j(a)\neq\emptyset \quad\text{and}\quad \left(\bigcup_1^k V_i(a)\right)\cap\left(\bigcup_1^k A_i\right)=\emptyset\,.$$

It is also clear that $cA_i^*\subset A_{i+1}^*$, $1\leqq i\leqq k-1$, and $cA_k^*\subset A_1^*$. The fact that $(A_1^*,\ldots,A_k^*)\gneq(A_1,\ldots,A_k)$ contradicts the maximality of $(A_1,\ldots,A_k)$.

2. $\{c^n a\}_1^\infty\cap\bigcup_1^k A_i\neq\emptyset$. In this case let $m\geqq 1$ be the first integer for which $c^m a\in\bigcup_1^k A_i$ and define $c^0a=a$. Then $c^m a\in A_{i_0}$, say, and $c^{m-1}a\notin\bigcup_1^k A_i$. Let A_i^*, $1\leqq i\leqq k$, be defined by

$$A_i^* = A_i \text{ if } i\neq i_0-1\,,$$
$$A_{i_0-1}^* = A_{i_0-1}\cup\{c^{m-1}a\}$$

in case $2\leqq i_0\leqq k$ and

$$A_i^* = A_i \text{ if } 1\leqq i\leqq k-1\,,$$
$$A_k^* = A_k\cup\{c^{m-1}a\}$$

in case $i_0=1$. It is clear that $cA_{i_0-1}^*\subset A_{i_0}^*$ in the first case and $cA_k^*\subset A_1^*$ in the second case and that A_i^*, $1\leqq i\leqq k$, are disjoint. Hence

$$(A_1^*,\ldots,A_k^*)\gneq(A_1,\ldots,A_k)$$

which contradicts again the maximality of $(A_1,\ldots,A_k)$. This shows that $V=\bigcup_1^k A_i$ which finishes the proof.

COROLLARY. *Let S be a right cancellation semigroup which contains an element c of infinite order. Then for any rational number $0\leqq r\leqq 1$ there exists a set $A\subset S$ such that $\varphi(A)=r$, for any c-left-invariant mean $\varphi\in m(S)^*$.*

PROOF. If $r=0$ or $r=1$ then $A=\emptyset$ or $A=S$ will satisfy the requirement. If $r=m/k$ where $0<m<k$ are integers then take $V=S$ in the above lemma. There are hence disjoint sets $A_1,\ldots,A_k$ such that

$$\bigcup_1^k A_i = S,\quad cA_i\subset A_{i+1} \text{ if } 1\leqq i\leqq k-1,\quad cA_k\subset A_1\,.$$

Hence, if φ is a c-left invariant mean, then

$$\varphi(A_{i+1}) \geqq \varphi(cA_i) \geqq \varphi(A_i) \quad \text{if } 1 \leqq i \leqq k-1 ,$$

and

$$\varphi(A_1) \geqq \varphi(cA_k) \geqq \varphi(A_k) .$$

Thus

$$\varphi(A_k) \leqq \varphi(A_1) \leqq \varphi(A_2) \leqq \dots \leqq \varphi(A_k) .$$

Therefore $\varphi(A_1)=\varphi(A_i)$ if $i \leqq k$ which shows that

$$k\varphi(A_1) = \sum_1^k \varphi(A_i) = \varphi(S) .$$

Therefore $\varphi(A_i)=1/k$ for $1 \leqq i \leqq k$, and $A=\bigcup_1^m A_i$ will satisfy $\varphi(A)=m/k=r$.

LEMMA 2. *Let S be a right cancellation semigroup and $c \in S$ an element of order $n \geqq 2$. Let $\emptyset \neq V \subset S$ be such that $cV \subset V$. Then there are n sets $A_1,\dots,A_n$ such that*

(i) $A_i \cap A_j = \emptyset$ *if* $i \neq j$,

(ii) $\bigcup_1^n A_i = V$,

(iii) $cA_i \subset A_{i+1}$ *if* $1 \leqq i \leqq n-1$ *and* $cA_n \subset A_1$.

PROOF. Let $a \in V$ and consider the sets $V_i=\{c^i a\}, 1 \leqq i \leqq n$. Then V_i are disjoint,

$$\bigcup_1^n V_i \subset V, \quad cV_i \subset V_{i+1} \text{ if } 1 \leqq i \leqq n-1, \quad cV_n \subset V_1 .$$

Let $\mathscr{V}$ be the set of n tuples $\{U_1,\dots,U_n\}$ such that $U_i \neq \emptyset$ are disjoint subsets of S,

$$\bigcup_1^n U_i \subset V, \quad cU_i \subset U_{i+1} \text{ if } 1 \leqq i \leqq n-1, \quad cU_n \subset U_1 .$$

We partially order $\mathscr{V}$ by

$$(U_1,\dots,U_n) \geq (V_1,\dots,V_n) \quad \text{iff} \quad V_i \subset U_i, \quad 1 \leqq i \leqq n .$$

Then $\mathscr{V} \neq \emptyset$ and as in the proof of lemma 1, $\mathscr{V}$ contains a maximal element say $(A_1,\dots,A_n)$. We show that $V=\bigcup_1^n A_i$. In fact if $a \in V$ and $a \notin \bigcup_1^n A_i$ then either

$$\{ca,\dots,c^n a\} \cap (\bigcup A_i) = \emptyset$$

or

$$\{ca,\dots,c^n a\} \cap (\bigcup A_i) \neq \emptyset .$$

In the first case the sets $A_i^*=A_i \cup \{c^i a\}$ would satisfy

$$(A_1^*,\dots,A_n^*) \in \mathscr{V} \quad \text{and} \quad (A_1^*,\dots,A_n^*) \gneqq (A_1,\dots,A_n)$$

which cannot be. In the second case let m be the first j for which

$1 \leqq j \leqq n$ and $c^j a \in \bigcup A_i$, and denote $c^0 a = a$. Then $c^m a \in A_{i_0}$, say. If $i_0 \geqq 2$ let

$$A^*_{i_0-1} = A_{i_0-1} \cup \{c^{m-1}a\}, \quad A_i{}^* = A_i \text{ if } i \neq i_0 - 1$$

and if $i_0 = 1$, let

$$A_n{}^* = A_n \cup \{c^{m-1}a\}, \quad A_i{}^* = A_i \text{ if } i \neq n \,.$$

In both cases

$$(A_1{}^*, \ldots, A_n{}^*) \in \mathscr{V} \quad \text{and} \quad (A_1{}^*, \ldots, A_n{}^*) \gneqq (A_1, \ldots, A_n) \,,$$

which cannot be. Hence $A_1, \ldots, A_n$ are the required sets.

Corollary 1. *Let S be a right cancellation semigroup $c \in S$ an element of order $n \geqq 2$. For any $r = k/n$, $k = 0, 1, \ldots, n$, there exists a set $A \subset S$ such that $\varphi(A) = r$ for any c-left invariant mean $\varphi \in m(S)^*$.*

Proof. If $r = 0$ or 1 then $\emptyset$ and S will satisfy the requirement. If $0 < r = k/n < 1$, let $A_1, \ldots, A_n$ be the sets of lemma 2. Then $A = \bigcup_1^k A_i$ will satisfy $\varphi(A) = k/n$.

Remark. Let G be the cyclic finite group of order n, $\{e, a, \ldots, a^{n-1}\}$, and φ the unique (a-) left invariant mean on G. Then $\varphi(A)$ ranges exactly over the values $\{k/n;\ k = 0, 1, \ldots, n\}$ when A ranges over all subsets of G. Hence Corollary 1 is the best possible result in this direction.

Corollary 2. *Let S be a right cancellation semigroup and $c \in S$. If a multiplicative c-leftinvariant mean $\varphi \in m(S)^*$ exists then c is an idempotent.*

Proof. If $A \subset S$ then $[\varphi(A)]^2 = \varphi(1_A \cdot 1_A) = \varphi(A)$. Hence $\varphi(A) = 0$ or 1. The corollary to lemma 1 implies that c has finite order n and corollary 1 of lemma 2 implies that $n = 1$, that is, $c^2 = c$.

Corollary 3. *Let S be a right cancellation* ELA *semigroup. Then S is the trivial group consisting of identity only.*

Proof. By corollary 2 of lemma 2, S contains only idempotents. Hence $ab^2 = ab$ for each $a, b \in S$ and so $ab = a$. This shows that if $a \neq b$ then $aS = a$ and $bS = b$ and so $aS \cap bS = \emptyset$ which cannot be. Hence S contains exactly one element e.

Corollary 4. *Let S be a semigroup, $S_0 \subset S$ a subsemigroup such that each two right ideals of S, and of S_0, have nonvoid intersection. If a S_0-left invariant multiplicative mean $\varphi \in m(S)^*$ exists then for each a, b in S_0 there is some $s \in S$ such that $as = bs$.*

Proof. We use as in Mitchell [14] a construction of [12]. For $a, b \in S$

we define $a(r)b$ iff there is some s in S such that $as=bs$. Then by the proof of lemma 2 of [12, p. 371] $a(r)b$ is a congruence relation (i.e. an equivalence relation such that $a\sim b$ implies $ca\sim cb$ and $ac\sim bc$ for each $c\in S$). If s' is the equivalence class which contains $s\in S$ then as in Ljapin [13, pp. 361–362] and [12, p. 371] $S'=\{s', s\in S\}$ becomes a right cancellation semigroup and $F\colon S\to S'$ defined by $Fs=s'$ is a homomorphism onto. Define now on $m(S')$ the following linear functional:

$$\varphi'(f') = \varphi(f'(F)) \quad \text{where} \quad (f'F)(s) = f'(F(s))$$

for $f'\in m(S')$ and s in S. It is clear that φ' is a multiplicative mean and since for each t in S and f' in $m(S')$,

$$(l_{s'}f')(F)(t) = f'(F(s)F(t)) = (f'F)(st) = (l_s(f'F))(t)\,,$$

where s is any representative of s', we have that φ' is $S_0'=\{s';\ s\in S_0\}$ left invariant on $m(S')$. Therefore $s_0'^2=s_0'$ for $s_0'\in S_0'$ and using the right cancellation one gets $s_1's_0'=s_1'$ for $s_1', s_0'\in S_0'$ which shows that $s_1'S_0'=s_1'$ for any $s_1'\in S_0'$. If now $s_1', s_0'\in S_0'$ then

$$s_1'S_0'\cap s_0'S_0' \neq \emptyset$$

which shows that S_0' contains exactly one element. This implies that $s_1\sim s_2$ for any $s_1, s_2\in S_0$ and therefore that $s_1s=s_2s$ for some $s\in S$.

We need in what follows the following observation due to Mitchell:

LEMMA (Mitchell [15, Cor. 6(a)]). *Let S be a semigroup such that for any $a_1, a_2\in S$, $a_1b=a_2b$ for some b in S. Then any two elements of S have a common right zero (i.e., if $a_1, a_2\in S$ then $a_1b=a_2b=b$ for some $b\in S$). Moreover if $A\subset S$ is finite then $Ac=c$ for some c in S.*

PROOF. There are c_1, c_2 such that $a_i^2c_i=a_ic_i$ and c such that

$$a_1c_1c = a_2c_2c = b\,.$$

Then $a_1b=a_2b=b$. Now by induction: If

$$A = \{a_1,\ldots,a_n\}\subset S, \quad a_ic_1 = c_1 \text{ for } 1\leqq i\leqq n-1\,,$$
$$a_nc_2 = c_2, \quad c_1c = c_2c = b\,,$$

then $a_ib=b$ for $1\leqq i\leqq n$.

THEOREM 1. *Let S be a* ELA *semigroup. Then any two elements of S have a common right zero.*

PROOF. S is in particular left amenable and therefore any two right ideals have nonvoid intersection. If $a, b\in S$, then by corollary 4, $ac=bc$ for some c in S and by Mitchell's lemma we can even assume that $ac=bc=c$.

This theorem has been proved by Mitchell [14] for the case that S is commutative or has left cancellation.

REMARKS. If S is a ELA semigroup and $a \in S$ then theorem 1 implies that $ab=b$ for some b in S. If S' is a representation of S as a semigroup of transformations on the set X under functional composition and if $s' \in S'$ then $s't'=t'$ for some t' and if $x_0 \in X$ then

$$s'(t'x_0) = (s't')(x_0) = t'(x_0) .$$

Hence $t'(x_0)$ is a fixed point for s' and so S has the individual fixed point property on any set (see Ljapin [13, p. 66]). It is this implication of theorem 1 which was conjectured to be false by Mitchell [14].

We are now after a Day–Følner type (see Day [4, p. 524–525] and Følner [9]) characterization of ELA semigroups.

LEMMA 3. *Let S be a semigroup. Then there exists a net $\{1_{s_\alpha}\}$ of point measures which converges in norm to left invariance if and only if any two elements of S have a common right zero.*

PROOF. Assume that any two elements of S have a common right zero. We introduce then a partial ordering in S (as in [12, p. 376]) which renders S a directed set (see definition in Kelley p. 65) as follows: We say that $b \geq a$ if either $b=a$ or $ab=b$. It is clear that $a \geq a$ and if $b \geq a$ and $c \geq b$ then

$$ac = a(bc) = (ab)c = bc = c ,$$

that is $c \geq a$ (for the case where $a \neq b$ and $b \neq c$; if either $a=b$ or $b=c$ then trivially $c \geq a$). Furthermore if $a,b \in S$, there is a c in S with $ac=bc=c$, that is $c \geq a$ and $c \geq b$. On the directed set $\{S, \geq\}$ define the net

$$\varphi_s = 1_s \in m(S)^* .$$

Let $a \in S$. Then

$$(L_a 1_{s_0})f = f(as_0) = 1_{as_0}f \quad \text{for } f \text{ in } m(S) .$$

Let $s_0 \in S$ be such that $as_0 = s_0$. If $s \geq s_0$ then, if $s_0 s = s$ it follows that $as = as_0 s = s_0 s = s$ and if $s = s_0$ it follows that $as = s$. Hence

$$\|L_a\varphi_s - \varphi_s\| = \|L_a 1_s - 1_s\| = \|1_{as} - 1_s\| = 0 \quad \text{for } s \geq s_0$$

which finishes the first part of the proof.

Conversely if 1_{s_α} converges in norm to left invariance and $a,b \in S$ then

$$\|L_a 1_{s_\alpha} - 1_{s_\alpha}\| < 1 \quad \text{and} \quad \|L_b 1_{s_\alpha} - 1_{s_\alpha}\| < 1$$

for $\alpha \geqq \alpha_0$, say. Now $\|1_a - 1_b\|$ is either 2 or 0 and is 0 if and only if $a=b$, as directly checked. Therefore $as_{\alpha_0} = s_{\alpha_0}$ and $bs_{\alpha_0} = s_{\alpha_0}$.

THEOREM 2 (strong extreme amenability). *Let S be a semigroup. Then S admits a multiplicative left invariant mean if and only if there exists a net of point measures $\{1_{s_\alpha}\}$ in $m(S)^*$ which converges in norm to left invariance.*

PROOF. If such a net $\{1_{s_\alpha}\}$ exists and φ_0 is any of its ω^* cluster points then as known (Day [4, pp. 520–521]) and directly checked, φ_0 is left invariant and multiplicative (since each 1_{s_α} is multiplicative). Conversely if S admits a multiplicative left invariant mean and $a,b \in S$ then $ac=bc$ for some $c \in S$ by theorem 1 and by Mitchell's lemma we can even assume that $ac=bc=c$. Invoke now the previous lemma.

REMARKS. Let $\beta(S) \subset m(S)^*$ be the set of all multiplicative means on $m(S)$ and consider $m(S)^*$ to be equipped with the Arens multiplication $\odot$ (Day [4, p. 527]), that is

$$\mu \odot \nu(f) = \mu(g) \quad \text{where} \quad g(s) = \nu(f_s) .$$

Then $\odot$ renders $\beta(S)$ (which as known is the Stone-Čech compactification of the discrete space S) a semigroup. In fact if $\mu,\nu \in \beta(S), f_1 f_2 \in m(S)$ and $g_i(s)=\nu(l_s f_i)$ then

$$(g_1 g_2)(s) = \nu\big(l_s(f_1 f_2)\big)$$

and therefore

$$\mu \odot \nu(f_1 f_2) = \mu(g_1 g_2) = \mu(g_1)\mu(g_2) .$$

S admits a multiplicative left invariant mean if and only if $\beta(S)$ contains a right zero, i.e. an element $\varphi_0 \in \beta(S)$ such that

$$\varphi \odot \varphi_0 = \varphi_0 \quad \text{for each} \quad \varphi \in \beta(S) .$$

If $\varphi \odot \varphi_0 = \varphi_0$ for $\varphi \in \beta(S)$, then (Day [4, p. 528, lemma 2])

$$L_s \varphi_0 = 1_s \odot \varphi_0 = \varphi_0 ,$$

hence φ_0 is left invariant. Conversely if φ_0 is left invariant and $f \in m(S)$ then $\varphi_0 f_s = \varphi_0 f$ and so $\varphi \odot \varphi_0 f = \varphi_0 f$ (Day [4, p. 530, cor. 4]).

We assemble now part of Mitchell's results together with those obtained in this paper to get the following characterization of ELA semigroups.

THEOREM 3 (characterization). *The following conditions on a semigroup S are equivalent:*

(1) *S has the common fixed point property on compacta.*

(2) *S admits a multiplicative left invariant mean.*

(3) *$\beta(S)$ has a right zero.*

(4) *There is a net* $\{1_{s_\alpha}\} \subset m(S)^*$ *of point measures, which converges in norm to left invariance.*

(5) *Each two elements of* S (*and hence each finite subset of* S) *have a common right zero.*

Condition (5) and hence each one of the above conditions implies (and is not in general implied by) that each element of S has a right zero (i.e. property $\prod$ of Ljapin [13, p. 66]).

REMARK. (1)$\Leftrightarrow$(2) and (5) $\Rightarrow$ (2) are due to Mitchell [14]. Furthermore due to Mitchell is also the implication (2) $\Rightarrow$ (5) under the additional assumption that S is either abelian or has left cancellation. (2) $\Rightarrow$ (5) in the present generality disproves Mitchell's conjecture in [14]. Our proof is entirely different from [14].

PROOF OF THEOREM 3. (2)$\Leftrightarrow$(3)$\Leftrightarrow$(4)$\Leftrightarrow$(5) has already been proved. Also, (1) $\Rightarrow$ (2) is immediate, using the fact that $\beta(S)$ is compact (and therefore there is a $\varphi_0 \in \beta(S)$ with $L_s\varphi_0 = \varphi_0$ for all s in S). We give a simple proof that (5) $\Rightarrow$ (1):

Let $\eta: S \to S'$ be a homomorphism of S onto a semigroup of continuous transformations from the compact space X into itself under functional composition, let $A \subset S$ be finite and

$$K_A = \{x \in X;\ (\eta s)(x) = x \text{ for each } s \in A\}.$$

K_A is closed and not empty since $Ac = c$ for some c in S and if $x_0 \in X$ and $s \in A$, then

$$\eta(s)[\eta(c)(x_0)] = \eta(sc)(x_0) = \eta(c)x_0,$$

that is $\eta(c)(x_0) \in K_A$. The closed sets $K_A \subset X$ have the finite intersection property and so $\bigcap_A K_A \neq \emptyset$ (where A runs over all finite subsets of S). Any $x_0 \in \bigcap_A K_A$ is a common fixed point for $\eta(S) = S'$.

In the semigroup consisting of the two elements $\{e_1, e_2\}$ with $e_ie_j = e_i$ for $1 \leqq i, j \leqq 2$ each element has a right zero (i.e. itself) but e_1, e_2 do not have a common right zero. For another example see Mitchell [14]. This finishes the proof.

REMARKS. The implications (3) $\Leftrightarrow$ (5) seem to us especially interesting in view of the trivial observation that S may satisfy (5) without having a right zero. In fact if $S = \{1, 2, 3, \ldots\}$ with $i \cdot j = \max\{i, j\}$ then each two elements of S have a common right zero while S does not. The same set S with usual addition or multiplication is not ELA and even moreover for any left invariant mean φ, $\varphi(A)$ has to take all rational values in $[0, 1]$, since the element 5 has infinite order for both considered operations.

3. The class of ELA semigroups.

PROPOSITION 1. (a) *If S is an* ELA *semigroup and S' a homomorphic image of S then S' is* ELA (*due to Mitchell* [14]).

(b) *If S is a semigroup with $S=\bigcup_{t\in T} S_t$ where S_t are* ELA *semigroups such that for any t_1,t_2 in T,*

$$S_{t_1} \cup S_{t_2} \subset S_{t_3} \quad \textit{for some} \quad t_3 \textit{ in } T ,$$

then S is ELA.

PROOF. (a) Any two elements of S have a common right zero and any homomorphic image of S enjoys this same property.

(b) If $a,b \in S$ then $a \in S_{t_1}$ and $b \in S_{t_2}$, hence $a,b \in S_{t_3}$ for some $t_3 \in T$. Hence $ac=bc=c$ for some $c \in S_{t_3} \subset S$.

REMARKS. As known, subsemigroups of LA semigroups need not be LA (Day [4, p. 516]). In analogy subsemigroups of ELA semigroups need not be even left amenable. In fact, let S_1 be the free *semigroup* on the two generators a,b. Let $S=S_1 \cup \{0\}$ with the multiplication given by $s0=0s=0$ if $s \in S_1$ and st is as in S_1 for $s,t \in S_1$. The semigroup S is ELA since it has a right zero while S_1 contains the two disjoint right ideals aS_1, bS_1, and hence is not even LA (as known).

Let S be a LA semigroup. If S_0 is a countable semigroup then S_0 is contained in a countable LA subsemigroup ([10, p. 50]). In analogy we have:

PROPOSITION 1a. *Let S be an* ELA *semigroup and $S_0 \subset S$ a countable subsemigroup. Then S_0 is contained in a countable* ELA *subsemigroup.*

PROOF. If $S_0=\{s_1,s_2,\ldots,s_k\ldots\}$, let $A_n=\{s_1,s_2,\ldots,s_n\}$. There are elements $c_2 \in S$ such that $A_2c_2=c_2$ and $c_3 \in S$ such that $B_3=A_3 \cup \{c_2\}$ satisfies $B_3c_3=c_3$. If $c_1,\ldots,c_{n-1}$ have been chosen, let c_n be such that $B_n= A_n \cup \{c_1,\ldots,c_{n-1}\}$ satisfies $B_nc_n=c_n$. Let S_0' be the countable semigroup generated by $\bigcup_1^\infty B_n$. Then $S_0 \subset S_0'$ and if $a,b \in S_0'$ then a,b are words in $a_1,\ldots,a_m$, where $\{a_1,\ldots,a_m\} \subset \bigcup_1^\infty B_n$. Since $B_n \subset B_{n+1}$,

$$\{a_1,\ldots,a_m\} \subset B_k \quad \text{for some } k .$$

Hence $a_ic_k=c_k$ for $i=1,2,\ldots,m$, and therefore $ac_k=bc_k=c_k$.

REMARKS. Let $\{S_t;\ t \in T\}$ be semigroups and $S=\prod_{t\in T} S_t$ be the set of all functions defined on T with $f(t) \in S_t$ for each $t \in T$. If $f,g \in S$, define the product $h=fg$ by $h(t)=f(t)g(t)$ (i.e. the product of $f(t)$ and $g(t)$ in S_t) for each t in T. The semigroup S is said to be the full direct product of $\{S_t;\ t \in T\}$.

Proposition 2. *If $\{S_t;\ t \in T\}$ are semigroups, then their full direct product $S=\prod_{t\in T} S_t$ is* ELA *if and only if each S_t is* ELA.

Proof. Since S_t are ELA, each two elements of S_t have a common right zero in S_t (by (2) $\Rightarrow$ (5) of the characterization theorem). If $f,g \in S$ and $h(t)$ is a common right zero of $f(t)$ and $g(t)$ in S_t, for each t in T, then h is a common right zero of f,g in S. Conversely, if any two elements of S have a common right zero then each S_t has this property.

Remarks. This proposition 2 breaks down the analogy between ELA semigroups and LA semigroups. As shown by M. Day [4, p. 517, F''] the full direct product of even amenable groups need not be amenable. We have emphasized in the above proof the point that in an ELA semigroup any two elements have a common right zero since it seems to us that a proof of proposition 2 using only the results given in Mitchell's paper [14] would be much more difficult.

Let S be a LA semigroup, φ a left invariant mean on $m(S)$ and $S_0 \subset S$ a subsemigroup with $\varphi(S_0)>0$. Then S_0 is LA. This is a result of Day [4, p. 518] and in analogy with it we prove:

Proposition 3. *Let S be a* ELA *semigroup, φ a multiplicative left invariant mean on $m(S)$ and $S_0 \subset S$ a subsemigroup with $\varphi(S_0)>0$. Then S_0 is* ELA.

Proof. Since $\varphi(A)$ is either 0 or 1, $\varphi(S_0)=1$. If $a,b \in S_0$ then $ad=bd=d$ for some d in S. Since $\varphi(dS)=\varphi(S_0)=1$, $dS \cap S_0 \neq \emptyset$. If $c \in dS \cap S_0$, then $ac=bc=c$ and $c \in S_0$.

We use in the next example the following trivial generalization of a result of Mitchell [14]:

Proposition 4. *Let S be a semigroup which contains a finite left ideal I. Then S is* ELA *if and only if S has a right zero. In particular* (*Mitchell* [14]) *if S is finite, then S is* ELA *if and only if S has a right zero.*

Proof. S can be represented as a semigroup of continuous transformations from the compact Hausdorff (discrete) I into itself by

$$s(i) = si \qquad \text{for} \quad s \in S \text{ and } i \in I.$$

If S is ELA, it has a common fixed point in I, i.e. there is some $i_0 \in I$ with $si_0 = i_0$ for each s in S. Conversely if S has a right zero then a fortiori any two elements of S have a common right zero.

Example 1. We give here an example of a semigroup S whose set of left invariant means is "huge", in fact it is even affinely homeomorphic with the set of all means on $m(S)$ and nevertheless this set does not touch any extreme point of the set of means in $m(S)^*$, that is, S is not ELA.

Let $S_\infty = \{e_1, e_2, \ldots, e_n, \ldots\}$ with $e_i e_j = e_j$ for each $1 \leqq i,j < \infty$ and let G be any finite group containing two or more elements. Let $S = G \times S_\infty$, i.e. the set of pairs (g, e_i) with $g \in G$ and

$$(g_1, e_i)(g_2, e_j) = (g_1 g_2, e_j) .$$

Then $\{(g, e_1);\ g \in G\}$ is a finite left ideal. If (g_0, e_j) would be a right zero of S then

$$(gg_0, e_j) = (g, e_i)(g_0, e_j) = (g_0, e_j)$$

for any g in G. Hence $gg_0 = g_0$ for $g \in G$ which cannot be since G is a nontrivial group. By proposition 4, the semigroup S is not ELA.

We show now that $A_i = \{(g, e_i);\ g \in G\}$, $i = 1, 2, 3, \ldots$, are the only groups and left ideals of S. In fact if A is a group and left ideal in S then it is a minimal left ideal and since it intersects some A_i we have $A \subset A_i$ for some i. But A_i are also minimal left ideals and therefore $A = A_i$. If $I = \{1, 2, 3, \ldots\}$ then $\{A_i;\ i \in I\}$ is the set of groups and left ideals of S. By [11, p. 102] there is an isometric isomorphism T from $m(I)^*$ *onto* the set of left invariant elements of $m(S)^*$ which maps the set of means of $m(I)^*$ *onto* the set of left invariant means of $m(S)^*$. Now the set of means in $m(I)^*$ is affinely homeomorphic to the set of means of $m(S)^*$, since S and I are both countable nonfinite. Hence the set of means of $m(S)^*$ is affinely homeomorphic with the set of left invariant means on $m(S)$.

Example 2. We give in what follows the promised example of a countable ELA semigroup S such that any of its subsemigroups which can be included in a finitely generated subsemigroup (and a fortiori any finitely generated subsemigroup) is not ELA, while the remaining subsemigroups, i.e. those which cannot be included in finitely generated subsemigroups are ELA. Furthermore each subsemigroup of S is LA and even admits an infinite dimensional set of left invariant means (none of which is multiplicative, if the subsemigroup can be included in a finitely generated one). Moreover S has left cancellation, is not right amenable and does not contain elements of finite order.

We shall need the following result due to A. H. Frey [8] (many thanks are due to A. H. Frey for kindly letting us have a preprint of his thesis) for which we give a new proof:

Lemma (A. H. Frey). *Let S be a semigroup and $I \subset S$ a left ideal. If I is* LA, *so is S. In particular, if I is commutative, then S is* LA.

PROOF. By Day's fixed point theorem ([5, p. 586]) there is a mean φ in $m(S)^*$ which is I-left invariant, that is $L_a\varphi = \varphi$ for $a \in I$. If now $s \in S$, choose some $a \in I$. Then

$$L_s\varphi = L_s(L_a\varphi) = L_{sa}\varphi = \varphi$$

since $sa \in I$. Hence φ is a left invariant mean on $m(S)$. If I is commutative then it is LA (see Day [2]).

Let S consist of all left sided sequences $s = (\ldots, s_n, \ldots, s_2, s_1)$ where $0 \leqq s_i < \infty$ are integers with only finitely many nonzero s_j and at least one $s_j \neq 0$ (i.e. the constant zero sequence is not included). For any s in S we define its degree, $d(s)$, as the biggest j for which $s_j \neq 0$ (Hence $d(s) \geqq 1$.) The multiplication in S is defined as follows: If $s,t \in S$ with $d(s) = m$ and $d(t) = n$ and

$$s = (\ldots, 0, s_m, s_{m-1}, \ldots, s_1), \quad t = (\ldots, 0, t_n, t_{n-1}, \ldots, s_1)\,,$$

then

$$st = \begin{cases} (\ldots, 0, t_n, \ldots, t_1) = t & \text{if } m = d(s) < d(t) = n\,, \\ (\ldots, 0, s_m, \ldots, s_{n+1}, s_n + t_n, t_{n-1}, t_{n-2}, \ldots, t_1) & \text{if } m = d(s) \geqq d(t) = n\,. \end{cases}$$

Hence $d(st) = \max\{d(s), d(t)\}$ and $s^k = (\ldots, 0, ks_n, s_{n-1}, \ldots, s_1)$, which shows that S contains only elements of infinite order. Furthermore, assuming that we have already shown associativity, we have that if $s^{(1)}, \ldots, s^{(k)}$ are elements in S which generate the subsemigroup S_0 and if $m = \max_{1 \leqq i \leqq k} d(s^{(i)})$ then $d(s) \leqq m$ for any s in S_0. This is deduced easily from $d(st) = \max\{(d(s), d(t))\}$ and $d(s^k) = d(s)$ for any $k \geqq 1$.

We show now that the multiplication is associative: Let $s,t,u \in S$, $d(s) = k$, $d(t) = m$, $d(u) = n$.

CASE 1. If $d(u) > \max(d(s), d(t)) = d(st)$, then $s(tu) = su = u$ since $d(u) > d(t)$, and $(st)u = u$ since $d(u) > d(st)$.

CASE 2. If $d(s) < d(t)$, then $(st)u = tu$ and since $d(s) < \max\{d(t), d(u)\} = d(tu)$ we have $s(tu) = tu$.

CASE 3. We can hence assume that $d(s) \geqq d(t)$ and $d(u) \leqq \max\{d(s), d(t)\} = d(st)$. Then $d(st) = d(s)$ and

$$(st)u = (\ldots, 0, s_k, s_{k-1}, \ldots, s_{m+1}, s_m + t_m t_{m-1}, \ldots, t_1)u$$

$$= \begin{cases} (\ldots, 0, s_k, s_{k-1}, \ldots, s_m + t_m, t_{m-1}, \ldots, t_{n+1}, t_n + u_n, u_{n-1}, \ldots, u_1) \\ \qquad \text{if } d(u) = n < m = d(t)\,, \\ (\ldots, 0, s_k, s_{k-1}, \ldots, s_{m+1}, s_m + t_m + u_m, u_{m-1}, \ldots, u_1) \\ \qquad \text{if } d(u) = n = m = d(t)\,, \\ (\ldots, 0, s_k, s_{k-1}, \ldots, s_{n+1}, s_n + u_n, u_{n-1}, \ldots, u_1) \\ \qquad \text{if } d(u) = n > m = d(t)\,. \end{cases}$$

Now

$$s(tu) = \begin{cases} s(\ldots,0,t_m,t_{m-1},\ldots,t_{n+1},t_n+u_n,u_{n-1},\ldots,u_1) & \text{if } d(u)=n<m=d(t) \\ s(\ldots,0,t_m+u_m,u_{m-1},\ldots,u_1) & \text{if } d(u)=n=m=dt \\ s(\ldots,0,u_n,u_{n-1},\ldots,u_1) & \text{if } d(u)=n>m=d(t) \end{cases}$$

$$= \begin{cases} (\ldots,0,s_k,s_{k-1},\ldots,s_{m+1},s_m+t_m,t_{m-1},\ldots,t_{n+1},t_n+u_n,u_{n-1},\ldots,u_1) \\ \qquad\qquad \text{if } n<m \\ (\ldots,0,s_k,s_{k-1},\ldots,s_{m+1},s_m+t_m+u_m,u_{m-1},\ldots,u_1) \quad \text{if } n=m \\ (\ldots,0,s_k,s_{k-1},\ldots,s_{n+1},s_n+u_n,u_{n-1},\ldots,u_1) \quad \text{if } n>m\,. \end{cases}$$

Thus $(st)u=s(tu)$ always.

Denote by $e^{(n)}$ that element of S which has 1 in the n'th place and 0 otherwise and let $ke^{(n)}$ be that element of S which has k in the n'th place and 0 otherwise. As directly checked the following relations hold:

$$(\ldots,0,s_n,s_{n-1},\ldots,s_1) = (s_ne^{(n)})(s_{n-1}e^{(n-1)})\ldots(s_1e^{(1)})$$

(if some $s_i=0$ then $s_ie^{(i)}$ does not appear),

$$(ne^{(i)})(me^{(i)}) = (n+m)e^{(i)}, \quad (ne^{(i)})(me^{(j)}) = (me^{(j)}) = me^{(j)} \text{ if } j>i\,.$$

Let now $s,t \in S$. If $m>\max\{d(s),d(t)\}$ then $se^{(m)}=e^{(m)}=te^{(m)}$ which shows that any two elements of S have a common right zero and so S is ELA.

Assume that S_0 is a subsemigroup which can be included in the subsemigroup generated by $\{s^{(1)},\ldots,s^{(k)}\}$. Then $d(s) \leqq \max_{1\leqq i\leqq k} d(s^{(i)})$ for any s in S_0. Let $m=\max\{d(s);\ s\in S_0\}$. Then $m=d(b)$ for some b in S_0. We claim that b^2 and b do not have a common right zero in S_0. Since if $c\in S_0$ is such that $b^2c=bc=c$ then

$$d(b) \geqq d(c) = \max\{d(c),d(b)\} \geqq d(b)\,.$$

Hence $d(b)=d(c)$ and so

$$bc = (\ldots,0,b_m+c_m,c_{m-1},\ldots,c_1) = (\ldots,0,c_m,c_{m-1},\ldots,c_1)\,.$$

Thus $b_m=0$, which contradicts the fact that $d(b)=m$. Hence S_0 is not ELA. That S has left cancellation is shown as follows:

Let $st=su$. If $d(s)<d(t)$ then $st=t$ and

$$d(t) = d(su) = \max\{d(s),d(u)\}\,.$$

Hence $d(u)>d(s)$. Therefore $su=u$ which shows that $t=u$. We can therefore assume that $d(s)\geqq d(t)$ and by symmetry that $d(s)\geqq d(u)$. If $d(s)=k$, $d(t)=m$, $d(u)=n$, then

$$(\ldots,0,s_k,\ldots,s_{m+1},s_m+t_m,t_{m-1},\ldots,t_1) \\ = (\ldots,0,s_k,\ldots,s_{n+1},s_n+u_n,u_{n-1},\ldots,u_1)\,.$$

If $m > n$, then $s_m + t_m = s_m$ and so $t_m = 0$ which cannot be. Hence $m \leqq n$ and by symmetry $n \leqq m$. Thus $m = n$ and

$$s_m + t_m = s_m + u_m, \quad t_{m-1} = u_{m-1}, \quad \ldots, \quad t_1 = u_1$$

which shows that $t = u$. We show now that every subsemigroup S_0 of S is left amenable and even moreover, has an infinite dimensional set of invariant means.

CASE 1. Assume that S_0 is generated by the finite set $\{t^{(1)}, \ldots, t^{(l)}\}$. Let $m = \max_{1 \leqq i \leqq l} d(t^{(i)})$ and let $\{s^{(1)}, \ldots, s^{(k)}\}$ be the set of all $t^{(i)}$'s of degree m. Let I_0 be the subsemigroup of S_0 consisting of all elements of the form

$$(\ldots, 0, n_1 s_m^{(1)} + n_2 s_m^{(2)} + \ldots + n_k s_m^{(k)}, s_{m-1}^{(1)}, s_{m-2}^{(1)}, \ldots, s_1^{(1)})$$

where $n_1, \ldots, n_k$ range over the set of nonnegative integers with $n_1 \geqq 1$, and

$$s^{(i)} = (\ldots, 0, s_m^{(i)}, s_{m-1}^{(i)}, \ldots, s_1^{(i)}) .$$

Since $n_1 \geqq 1$, the degree of any element of I_0 is m. Furthermore since

$$\begin{aligned}(\ldots, 0, a_m, a_{m-1}, \ldots, a_1)(\ldots, 0, b_m, a_{m-1}, \ldots, a_1) \\ &= (\ldots, 0, a_m + b_m, a_{m-1}, \ldots, a_1) \\ &= (\ldots, 0, b_m, a_{m-1}, \ldots, a_1)(\ldots, 0, a_m, a_{m-1}, \ldots, a_1)\end{aligned}$$

(if $a_m \neq 0$ and $b_m \neq 0$) we have that I_0 is a commutative semigroup. Furthermore

$$\begin{aligned}(\ldots, 0, n_1 s_m^{(1)} + n_2 s_m^{(2)} + \ldots + n_k s_m^{(k)}, s_{m-1}^{(1)}, s_{m-2}^{(1)}, \ldots, s_1^{(1)}) \\ &= (s^{(k)})^{n_k} (s^{(k-1)})^{n_{k-1}} \ldots (s^{(2)})^{n_2} (s^{(1)})^{n_1})\end{aligned}$$

where $(s^{(i)})^j$ is $s^{(i)}$ to the j-th power and $(s^{(i)})^0$ means that $s^{(i)}$ does not appear. Hence $I_0 \subset S_0$. But I_0 is even a left ideal of S_0. In fact if $t^{(i)}$ is such that $d(t^{(i)}) < m$ then $t^{(i)} s = s$ for any s in I_0. If $d(t^{(i)}) = m$, then $t^{(i)} = s^{(j)}$ for some j and therefore

$$\begin{aligned}(\ldots, 0, s_m^{(j)}, s_{m-1}^{(j)}, \ldots, s_1^{(j)}) \\ (\ldots, 0, n_1 s_m^{(1)} + \ldots + n_j s_m^{(j)} + \ldots + n_k s_m^{(k)}, s_{m-1}^{(1)}, s_{m-2}^{(1)}, \ldots, s_1^{(1)}) \\ = (\ldots, 0, n_1 s_m^{(1)} + \ldots + (n_j + 1) s_m^{(j)} + \ldots + n_k s_m^{(k)}, s_{m-1}^{(1)}, s_{m-2}^{(1)}, \ldots, s_1^{(1)})\end{aligned}$$

which belongs to I_0. Since $s I_0 \subset I_0$ holds true for any generator of S_0 we get that $S_0 I_0 \subset I_0$, that is, I_0 is a left ideal of S_0 and I_0 is commutative. This implies, by Frey's lemma, that S_0 is left amenable.

CASE 2. If S_0 is not finitely generated then $S_0 = \{s^{(1)}, s^{(2)}, \ldots, s^{(n)}, \ldots\}$. If S_n is the subsemigroup of S_0 generated by $\{s^{(1)}, \ldots, s^{(n)}\}$ then $S_0 =$

$\bigcup_1^\infty S_n$ and $S_n \subset S_{n+1}$. By case 1 each S_n is left amenable and by Day [3] or Dixmier [6, p. 215–216], S_0 is left amenable. The set of left invariant means on $m(S_0)$ is infinite dimensional since its finite dimensionality would imply by [10, p. 56] that S_0 is finite. The fact that S does not contain elements of finite order furnishes the desired contradiction.

Now S is not right amenable since it contains the two disjoint left ideals $Se^{(1)}$ and $Se^{(2)}$. ($Se^{(1)}$ is the set of all the sequences s in S for which $s_1 \geqq 1$ while $Se^{(2)}$ is the set of all the sequences s in S for which $s_2 \geqq 1$ and $s_1 = 0$). In fact moreover, if S_0 is a subsemigroup of S which contains two elements s,t of degree m,n resp. such that if $k = \min(m,n)$ then the $k-1$ tuples $(s_{k-1},\ldots,s_1)$, $(t_{k-1},\ldots,t_1)$ are not equal, then S_0 is not right amenable (since in this case $S_0 s$, $S_0 t$ are two disjoint left ideals of S_0).

If S_0 is a subsemigroup which cannot be included in any finitely generated subsemigroup of S then for any n_0 there is some s_0 in S_0 with $d(s_0) > n_0$. Otherwise if $s \in S_0$ then $d(s) = m \leqq n_0$ and therefore

$$s = (\ldots, 0, s_m, s_{m-1}, \ldots, s_1) = (s_m e^{(m)})(s_{m-1} e^{(m-1)}) \ldots (s_1 e^{(1)}) .$$

Hence S_0 is included in the subsemigroup generated by

$$\{e^{(1)}, e^{(2)}, \ldots, e^{(n_0)}\}$$

which cannot be. If now $s,t \in S_0$ and $n_0 = \max\{d(s), d(t)\}$ then let $u \in S_0$ be such that $d(u) > n_0$. Then $su = tu = u$ which implies that S_0 is ELA.

Remark. After constructing the example used in the preceding proof we found out that this same example S has been considered before, by E. S. Ljapin, in a different context. E. S. Ljapin has shown that the semigroup S plays a particularly important role in the structure theory of the semigroups with the property that each element has a right zero (i.e. with the individual fixed point property on any set). See Ljapin [13, p. 69, p. 336 and pp. 339–344]. It seems to us that even the proof of the associativity of multiplication given above is still not superfluous, since our representation of S is somewhat different from that of Ljapin.

Remarks. In the above example 2 of the ELA semigroup S each finitely generated subsemigroup S_0 is not ELA. This happens since S does not contain "enough" (in fact not at all) periodic elements. We have:

Proposition 6. *Let S be an* ELA *semigroup such that each right ideal of S contains a periodic element. If S_0 is a subsemigroup generated by the finite set $\{s_1, \ldots, s_n\}$ then there is some s_{n+1} in S such that the subsemigroup generated by $\{s_1, \ldots, s_n, s_{n+1}\}$ is* ELA.

Proof. Let $c \in S$ be such that $s_i c = c$ for $i = 1, 2, \ldots, n$. Then there is some $d \in cS$ such that $e = d^k$ satisfies $e^2 = e$ and $s_i e = e$ (since $e \in cS$). Let S_0' be the semigroup generated by $\{s_1, \ldots, s_n, e\}$. Then S_0' has a right zero, namely e and is hence a fortiori ELA. (An element $c \in S$ is periodic iff it generates a finite subsemigroup. If S has left or right cancellation, then $c \in S$ is periodic iff c has finite order.)

Proposition 7. *Let S be a semigroup and $I \subset S$ a left ideal. Then S is* ELA *if and only if I is* ELA.

Proof. If S is ELA, $s \in S$, $\varphi \in m(I)^*$ and $f \in m(I)$, then define $(f_s)(a) = f(sa)$ for any a in I. Since I is a left ideal $f_s \in m(I)$. Define now $(L_s\varphi)f = \varphi(f_s)$. Then L_s are ω^*-continuous on $m(I)^*$ and map $\beta(I)$ into itself. Hence by Mitchell's fixed point theorem ((1) $\Leftrightarrow$ (2) of the characterization theorem) there is an element $\varphi_0 \in \beta(I)$ with $L_s\varphi_0 = \varphi_0$ for $s \in S$ and a fortiori for s in I.

Conversely, if I is ELA then again by Mitchell's fixed point theorem there is some $\varphi_0 \in \beta(S)$ with $L_a\varphi_0 = \varphi_0$ for each a in I. If $s \in S$, let $a \in I$ be arbitrary. Then

$$L_s\varphi_0 = L_s(L_a\varphi_0) = L_{sa}\varphi_0 = \varphi_0 ,$$

since $sa \in I$.

Remark. Replacing in the above proof, $\beta(I)\,[\beta(S)]$ by the set of means on $m(I)\,[m(S)]$ and Mitchell's fixed point theorem by Day's fixed point theorem (see [5]) one gets that if S is a semigroup and $I \subset S$ a left ideal then S is LA if and only if I is LA. The "if" part is due to A. H. Frey [8] and the "only if" part to Mitchell [15].

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