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# A THEOREM ON THE MAXIMUM MODULUS OF ENTIRE FUNCTIONS

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Let $f(z)$ be an entire function. We denote $\max|f(z)|$ and $\min|f(z)|$ on $|z|=r$ by $M(r)$ and $m(r)$, respectively. Let $\lambda$ satisfy $0<\lambda<1$. We shall study $r^{-\lambda}\log M(r)$ for large values of $r$. We assume that $f(0)=1$, which is an unessential restriction.

We define

$$I(R) = \int_0^R r^{-1-\lambda}\bigl(\log m(r) - \cos\pi\lambda \log M(r)\bigr)\, dr .$$

If $\log M(r) = O(r^\lambda)$, then $f(z)$ is at most of order $\lambda$, and we have the following representation:

$$f(z) = \prod_1^\infty (1 - z/z_n) .$$

We form an auxiliary function

$$f_1(z) = \prod_1^\infty (1 + z/|z_n|)$$

and denote $\max|f_1(z)|$ and $\min|f_1(z)|$ on $|z|=r$ by $M_1(r)$ and $m_1(r)$, respectively. We also need

$$I_1(R) = \int_0^R r^{-1-\lambda}\bigl(\log m_1(r) - \cos\pi\lambda \log M_1(r)\bigr)\, dr .$$

Our main result is

THEOREM 1. *Let $f(z)$ be an entire function and let $\lambda$ be any number satisfying $0<\lambda<1$.*

A. *If the finite limit $\lim_{r\to\infty} r^{-\lambda}\log M(r)$ exists, then $\lim_{R\to\infty} I(R)$ exists, but this limit may be infinity. If the finite limit $\lim_{R\to\infty} I(R)$ exists, then $\lim_{r\to\infty} r^{-\lambda}\log M(r)$ exists, but this limit may be infinity.*

B. *If $r^{-\lambda}\log M(r)$ and $I(R)$ have an upper bound, then $I(R)$ is bounded below, and $\lim_{r\to\infty} r^{-\lambda}\log M(r) = \alpha$ exists if and only if $\lim_{R\to\infty} I(R)$ exists.*

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*If these limits exist, then the limits* $\lim_{r\to\infty} r^{-\lambda}\log M_1(r)=\beta$ *and* $\lim_{R\to\infty} I_1(R)$ *also exist, and*

$$\alpha = \beta = -\lim_{R\to\infty} I_1(R)/(\pi \sin \pi\lambda) .$$

Many papers have been devoted to problems of this type. For a list of references, the reader is referred to Kjellberg [7]. In this paper, Kjellberg uses methods from the theory of analytic functions to prove the following result.

THEOREM 2. *Let* $f(z)$ *be an entire function and* $\lambda$ *any number satisfying* $0<\lambda<1$. *If for all sufficiently large values of* $r$,

$$\log m(r) - \cos \pi\lambda \log M(r) \leqq 0 ,$$

*then* $\lim_{r\to\infty} r^{-\lambda}\log M(r)$ *exists. The limit is positive or infinite.*

REMARK. The existence of the limit also follows from the weaker assumption that $\log m(r) - \cos\pi\lambda \log M(r)$ has an upper bound. The proof is the same.

In [3], Essén gave an alternative proof of this theorem using general properties of integral inequalities. A conjecture by Kjellberg that these methods could yield further results has led to the present paper.

Recently Anderson [1] has used methods from the theory of analytic functions to prove the following theorem:

THEOREM 3. *Suppose* $f(z)$ *is an integral function and* $\lambda$ *a fixed number satisfying* $0<\lambda<1$. *If*

$$\alpha = \underline{\lim}_{r\to\infty} r^{-\lambda}\log M(r) < \infty$$

*and*

$$\overline{\lim}_{r_1, r_2\to\infty} I(r_1,r_2) \leqq 0$$

*where* $I(r_1,r_2)=I(r_2)-I(r_1)$, $r_1<r_2$, *then*

$$\log M(r) \sim \alpha r^{\lambda} \quad (r\to\infty) .$$

*Moreover*

$$\log M_1(r) \sim \alpha r^{\lambda} \quad (r\to\infty) .$$

Before proving Theorem 1, we want to show that Theorem 3 and the part of Theorem 2 which deals with the existence of $\lim_{r\to\infty} r^{-\lambda}\log M(r)$ are consequences of Theorem 1 and the following simple reformulation of results by Kjellberg [6]:

LEMMA 1. *Let* $f(z)$ *be an entire function and let* $\lambda$ *be any number satisfying* $0<\lambda<1$. *If* $I(R)$ *has an upper bound and* $\underline{\lim}_{r\to\infty} r^{-\lambda}\log M(r)<\infty$, *then* $\overline{\lim}_{r\to\infty} r^{-\lambda}\log M(r)<\infty$.

PROOF OF THE EXISTENCE OF $\lim_{r\to\infty} r^{-\lambda}\log M(r)$ IN THEOREM 2. If $\log m(r) - \cos\pi\lambda \log M(r)$ has an upper bound, the same is true for $I(R)$. It follows from Lemma 1 that either $\lim_{r\to\infty} r^{-\lambda}\log M(r) = \infty$ or $\log M(r) = O(r^{\lambda})$. In the first case, we have nothing to prove. In the second case, we have, according to Theorem 1, that $I(R)$ is bounded below. Since $\log m(r) - \cos\pi\lambda \log M(r)$ has an upper bound, it follows that $\lim_{R\to\infty} I(R)$ exists, and Theorem 1 now gives the existence of $\lim_{r\to\infty} r^{-\lambda}\log M(r)$.

PROOF OF THEOREM 3. From the assumptions, it follows by arguments analogous to those above that $\log M(r) = O(r^{\lambda})$ and that $\lim_{R\to\infty} I(R)$ exists and is finite, and Theorem 1 gives the existence of the limits.

In [6], Kjellberg mentioned that the results in his paper could have been stated for subharmonic functions. This is also true for the results in [7] and the results in the present paper. In [2], Anderson has, among other things, made this generalization of his results in [1].

Several lemmas are needed in the proof of Theorem 1. In these lemmas, $\lambda$ always denotes a number satisfying $0 < \lambda < 1$.

LEMMA 2. *If* $\log M_1(r) = O(r^{\lambda})$, *then* $I_1(R)$ *is bounded.*

LEMMA 3. *If* $\log M_1(r) = O(r^{\lambda})$, *then* $\lim_{r\to\infty} r^{-\lambda}\log M_1(r) = \beta$ *exists if and only if* $\lim_{R\to\infty} I_1(R) = \gamma$ *exists. If these limits exist, then*

$$\pi\beta \sin\pi\lambda = -\gamma .$$

REMARK. The proof of Lemma 2 is similar to the proof of Theorem 5 in Boas [3] and to the beginning of the proof of Theorem 1 in Anderson [1]. Results such as those in Lemma 3 are well known and usually proved with methods from the theory of analytic functions. As an instance, we can mention Theorem 1 in Anderson [1]. We give a new proof, using the convolution equation which we deduce from formula (15) in Kjellberg [7].

LEMMA 4. a) *If* $I_1(R)$ *is bounded, then either* $I(R)$ *is bounded* $\lim_{R\to\infty} I(R) = \infty$.

b) *If* $\log M(R) = O(R^{\lambda})$ *and if* $I(R)$ *has an upper bound, then* $\lim_{R\to\infty} I(R)$ *exists if and only if* $\lim_{R\to\infty} I_1(R)$ *exists.*

PROOF OF LEMMA 2. We consider $z^{-1-\lambda}\log f_1(z)$, defined so as to be real on the positive real axis, and we integrate over a half-circle in the upper half-plane and a diameter on the real line in the same way as in Section 3 in Anderson [1]. Since $\log M_1(r) = O(r^{\lambda})$, we obtain

$$\int_0^R x^{-1-\lambda}[\log f_1(x) - e^{-i\lambda\pi}(\log|f_1(-x)| + i\pi n(x))]\,dx = O(1)\,.$$

Here $n(x)$ is the number of zeros of $f_1(z)$ in $|z| \leqq x$, counted with multiplicities. Taking real parts after multiplication by the factor $-e^{i\lambda\pi}$, we obtain the lemma.

In the proofs of Lemmas 3 and 4, we need much additional notation. We start from the formula (15) in Kjellberg [7] and the convolution equation which is obtained from (15) by the change of variables $R = e^x$ and $r = e^y$ (cf. Essén [5]). We define

$$\begin{aligned}
\varphi(x) &= e^{-\lambda x}\log M(e^x)\,,\\
\eta(x) &= e^{-\lambda x}\log m(e^x)\,,\\
\varphi_1(x) &= e^{-\lambda x}\log M_1(e^x)\,,\\
\eta_1(x) &= e^{-\lambda x}\log m_1(e^x)\,,\\
p(x) &= \eta(x) - \cos\pi\lambda\,\varphi(x)\,,\\
P(x) &= \int_0^x p(t)\,dt\,,\\
p_1(x) &= \eta_1(x) - \cos\pi\lambda\,\varphi_1(x)\,,\\
P_1(x) &= \int_0^x p_1(t)\,dt\,,\\
K(x) &= \frac{2(1+\cos\pi\lambda)\,x\,e^{x(1-\lambda)}}{\pi^2(e^{2x}-1)}\,.
\end{aligned}$$

It is clear that

$$I(e^x) - I(1) = P(x) \quad \text{and} \quad I_1(e^x) - I_1(1) = P_1(x)\,.$$

We also have that

$$\lim_{x\to-\infty}\varphi(x) = \lim_{x\to-\infty}\varphi_1(x) = 0\,.$$

On all occasions when the following formulas are used, the functions $\varphi$ and $\varphi_1$ will be bounded.

(1) $$\eta_1 \leqq \eta \leqq \varphi \leqq \varphi_1\,,$$

(2) $$\eta_1 + \varphi_1 \leqq \eta + \varphi\,,$$

(3) $$p_1 \leqq p\,,$$

(4) $$\varphi_1(1+\cos\pi\lambda) = \eta_1 * K + \varphi_1 * K\,.$$

The formula (1) is well known. The formulas (2) and (4) are obtained from (13) and (15) in Kjellberg [7] by the change of variables mentioned above. Also, the formula (3) is known, but we have not found any explicit reference. The formula is valid since

$$\begin{aligned} p_1 = \eta_1 - \varphi_1 \cos\pi\lambda &= \eta_1 + \varphi_1 - \varphi_1(1+\cos\pi\lambda) \\ &\leqq \eta + \varphi - \varphi(1+\cos\pi\lambda) = p\,. \end{aligned}$$

In the inequality, we have used (2) and the right inequality in (1). In (4), we eliminate $\eta_1$ and obtain

$$\varphi_1 - \varphi_1 * K = \frac{p_1 * K}{1+\cos\pi\lambda}\,. \tag{5}$$

If we express $\eta$ and $\eta_1$ in terms of $p$, $p_1$, $\varphi$ and $\varphi_1$ and use (4), we obtain

$$\varphi_1(1+\cos\pi\lambda) + p_1 \leqq \varphi(1+\cos\pi\lambda) + p\,. \tag{6}$$

In the proof of Theorem 1, the formulas (5) and (6) are essential.

In the proof of Lemma 3, we need two more lemmas.

Lemma 5. *Let $t$ be a real number. Then*

$$\hat{K}(t) = \int_{-\infty}^{\infty} e^{-itx} K(x)\,dx \neq 0\,.$$

Proof. A residue calculation shows that

$$\hat{K}(t) = \frac{2(1+\cos\pi\lambda)}{\{e^{\frac{1}{2}\pi(t-i\lambda)} + e^{-\frac{1}{2}\pi(t-i\lambda)}\}^2}$$

and hence the lemma is true.

Lemma 6. *We define*

$$N(x) = \begin{cases} \displaystyle\int_x^{\infty} K(y)\,dy & x > 0\,, \\ \displaystyle\int_{-\infty}^{-x} K(y)\,dy & x < 0\,. \end{cases}$$

*Then $\hat{N}(t) \neq 0$ for all real values of $t$.*

Proof. See Essén [5].

Proof of Lemma 3. We shall use Pitt's Tauberian theorem (Th. 10 a, ch. V in [8]) in the same way as in Essén [5]. We shall use the fact that $\varphi$ and $\varphi_1$ are slowly decreasing (cf. Def. 9b, ch. V in [8] for the definition and Essén [5] for the proof).

Integrating (5) and applying an integration by parts, we get

$$\varphi_1 * N(x) - \varphi_1 * N(0) = \frac{1}{1+\cos\pi\lambda}\big(P_1 * K(x) - P_1 * K(0)\big)\,. \tag{7}$$

It is clear that the following proposition is true:

$$\lim_{x\to\infty}\varphi_1 * N(x) \text{ exists} \quad \text{iff} \quad \lim_{x\to\infty} P_1 * K(x) \text{ exists}\,. \tag{8}$$

We have assumed that $\varphi_1$ is bounded and it follows from Lemma 2 that $P_1$ is bounded. We know, according to the Lemmas 5 and 6 that the value 0 is not assumed by $\hat{N}(t)$ and $\hat{K}(t)$. As just mentioned, we also know that $\varphi_1$ is slowly decreasing. If $P_1$ fulfills a Tauberian condition, for instance that $P_1$ is slowly increasing, we can use (8) and Pitt's Tauberian theorem and prove that $\lim_{x\to\infty}\varphi_1(x) = \beta$ exists if and only if $\lim_{x\to\infty} P_1(x) = \gamma$ exists.

It remains to prove that $P_1$ is slowly increasing and that $\beta$ has the value given in the lemma. Since, if $h$ is a positive number,

$$\begin{aligned} P_1(x+h) - P_1(x) &= \int_x^{x+h} \big(\eta_1(y) - \cos\pi\lambda\,\varphi_1(y)\big)\,dy \\ &\leqq \int_x^{x+h} (1-\cos\pi\lambda)\,\varphi_1(y)\,dy \\ &\leqq \log M_1(e^{x+h}) \int_x^{x+h} (1-\cos\pi\lambda)\,e^{-\lambda y}\,dy \\ &= \varphi_1(x+h)\,(1-\cos\pi\lambda)\,(e^{-\lambda h}-1)\,\lambda^{-1}\,, \end{aligned}$$

and since $\varphi_1$ is bounded, it follows that

$$\varlimsup_{h\to+0}\ \varlimsup_{x\to\infty}\big(P_1(x+h) - P_1(x)\big) \leqq 0\,.$$

This means that $P_1$ is slowly increasing.

Since $\lim_{x\to-\infty}\varphi_1(x) = 0$ and $\lim_{x\to-\infty} P_1(x) = 0$, it follows from (7) that

$$\varphi_1 * N(0)\,(1+\cos\pi\lambda) = P_1 * K(0)\,,$$

and hence

$$\varphi_1 * N(x)\,(1+\cos\pi\lambda) = P_1 * K(x)\,.$$

If the limits exist, we obtain that

$$\beta\hat{N}(0)\,(1+\cos\pi\lambda) = \gamma\cdot\hat{K}(0)\,,$$

and since $\hat{N}(0) = -\pi\,\mathrm{tg}(\pi\lambda/2)$ and $\hat{K}(0) = 1$ (cf. Essén [5]), it follows that $\beta\pi\sin\pi\lambda = -\gamma$. The lemma is proved.

PROOF OF LEMMA 4. We know that $P(x)=I(e^x)-I(1)$ and that $P_1(x)=I_1(e^x)-I_1(1)$, and hence it suffices to prove the corresponding statements for $P$ and $P_1$. Consider the following equation:

$$P(x) = P_1(x) + \int_0^x (p(t)-p_1(t))\,dt\,. \tag{9}$$

Since, according to (3), the integrand is positive, we have that the integral has a finite or infinite limit as $x \to \infty$. Hence a) is true.

If $\log M(r)=O(r^\lambda)$, it is also true that $\log M_1(r)=O(r^\lambda)$ (cf. Boas [4, Theorem 2.9.5]). It now follows from Lemma 2 that $I_1$ is bounded. Hence $P_1$ is bounded and b) follows from (9).

PROOF OF THEOREM 1. We shall use the notation given after the proof of Lemma 2. In order to simplify the reading of the proof, we state the lemmas once more, now using the new notation.

LEMMA 1. *If $P$ has an upper bound and $\underline{\lim}_{x\to\infty}\varphi(x)$ is finite, then $\overline{\lim}_{x\to\infty}\varphi(x)$ is finite.*

LEMMA 2. *If $\varphi_1$ is bounded, then $P_1$ is bounded.*

LEMMA 3. *If $\varphi_1$ is bounded, then $\lim_{x\to\infty}\varphi_1(x)=\beta$ exists if and only if $\lim_{x\to\infty}P_1(x)=\gamma$ exists. If these limits exist, then $\pi\beta\sin\pi\lambda=-\gamma$.*

LEMMA 4. a) *If $P_1$ is bounded then we have that either $P$ is bounded or $\lim_{x\to\infty}P(x)=\infty$.*

b) *If $\varphi$ is bounded and if $P$ has an upper bound, then $\lim_{x\to\infty}P(x)$ exists if and only if $\lim_{x\to\infty}P_1(x)$ exists.*

We first prove that if $\lim_{r\to\infty}r^{-\lambda}\log M(r)=\lim_{x\to\infty}\varphi(x)$ exists and is finite, then $\lim_{R\to\infty}I(R)=\lim_{x\to\infty}P(x)$ exists. The latter limit may be infinite.

Assume that $\lim_{x\to\infty}\varphi(x)$ exists and is finite. Since $\varphi$ is bounded, it then follows (as mentioned in the proof of Lemma 4b) that $\varphi_1$ is bounded and hence, according to Lemma 2, that $P_1$ is bounded. Now, Lemma 4a implies that either $\lim_{x\to\infty}P(x)=\infty$ or $P$ is bounded. In the first case, there is nothing more to prove. If $P$ is bounded, we integrate (6) and obtain

$$(1+\cos\pi\lambda)\int_0^x (\varphi_1(y)-\varphi(y))\,dy \leqq P(x)-P_1(x)\,. \tag{10}$$

Since the right member is finite and since, according to (1) $\varphi_1-\varphi$ is non-negative, $\int_0^\infty(\varphi_1(t)-\varphi(y))dy$ is convergent. Now, $\varphi_1$ is slowly decreasing (cf. the beginning of the proof of Lemma 3) and $\lim_{x\to\infty}\varphi(x)$ exists, and hence $\varphi_1-\varphi$ is slowly decreasing. But if the integral of a positive, slowly decreasing function $g$ is convergent at infinity, we must have that $\lim_{x\to\infty}g(x)=0$. Hence

$$\lim_{x\to\infty}\varphi_1(x) = \lim_{x\to\infty}\varphi(x) = \alpha .$$

Since $\lim_{x\to\infty}\varphi_1(x)$ exists and is finite, Lemma 3 implies that $\lim_{x\to\infty}P_1(x)$ exists and is finite, and gives the value of $\alpha$. Since $P$ is bounded and $\lim_{x\to\infty}P_1(x)$ exists, the existence of $\lim_{x\to\infty}P(x)$ follows from Lemma 4b. Hence we have proved that the existence of $\lim_{x\to\infty}\varphi(x)$ implies the existence of $\lim_{x\to\infty}P(x)$, e.g. the first part of A in Theorem 1. We have also proved the "only if"-part of B in Theorem 1.

Conversely, let us assume that $\lim_{x\to\infty}P(x)$ exists and is finite. It follows from Lemma 1 that either $\lim_{x\to\infty}\varphi(x)=\infty$ or $\varphi$ is bounded. In the first case, there is nothing more to prove. In the second case, $\varphi$ and hence $\varphi_1$ are bounded, and Lemma 2 implies that $P_1$ is bounded. Now we first use Lemma 4b to conclude that $\lim_{x\to\infty}P_1(x)$ exists, and secondly we use Lemma 3 to conclude that $\lim_{x\to\infty}\varphi_1(x)$ exists. Since $\varphi$ is slowly decreasing, it follows that $\varphi_1-\varphi$ is slowly increasing. With the same kind of argument as above, it follows from (10) that $\lim_{x\to\infty}\varphi(x)$ exists. Hence the converse statements are proved and the proof is complete.

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