A MEASURE THEORETIC CHARACTERIZATION OF CHOQUET SIMPLEXES

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A point set $A$ in $\mathbb{R}^n$ is affinely independent if and only if the convex hulls of $B$ and $C$ are disjoint, hence separable by a hyperplane, for any partition $\{B, C\}$ of $A$. (This is related to a classical theorem of Radon, cf. e.g. [4].) In the present paper this result is generalized to a theorem on a compact convex set $K$ in a locally convex space, according to which $K$ is a simplex if and only if every boundary measure on $K$ admits a Hahn decomposition by halfspaces (determined by an affine Borel function of class $\mathcal{A}$, cf. definition below).

We are indebted to R. Phelps for valuable discussions on the subject and also for making available to us the manuscript of his forthcoming book [6].

1. Definitions and basic properties.

The setting of the present note is similar to that of [1], and we shall use the concepts of that paper rather freely. Thus $K$ shall be a compact convex subset of a locally convex Hausdorff vector space $E$ over $\mathbb{R}$, $\mathcal{K}$ shall be the class of all $K$-restrictions of continuous, affine functionals, and $\mathcal{S}$ shall be the class of all continuous and convex, real valued functions on $K$. The lower envelope $\tilde{f}$ of a real valued function $f$, bounded below on $K$, is the greatest l.s.c. convex minorant of $f$. It can be expressed as follows

$$\tilde{f}(x) = \sup \{g(x) \mid g \in \mathcal{S}, \ g(y) < f(y) \text{ for all } y \in K\}$$

$$= \sup \{h(x) \mid h \in \mathcal{K}, \ h(y) < f(y) \text{ for all } y \in K\}.$$

The upper envelope $\bar{f}$ is defined dually and admits the dual characterizations.

In the sequel we shall use the word measure to denote a regular Borel measure on $K$, and vector-valued integrals are taken in the weak sense. Thus $\int_{\mathbb{R}} f \, d\mu(t)$ denotes the resultant of $\mu$, and it denotes the barycenter of $\mu$ if $\mu$ is positive and normalized (probability measure).

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We shall make repeated use of the following elementary fact: If \( \{ f_\alpha \} \) is an ascending net of l.s.c. functions such that \( \sup_\alpha \int f_\alpha \, d\mu < \infty \) for some measure \( \mu \), then the l.s.c. function \( f = \sup_\alpha f_\alpha \) is \( \mu \)-integrable and

\[
\int f \, d\mu = \sup_\alpha \int f_\alpha \, d\mu .
\]

A measure \( \mu \) is said to be a \textit{boundary measure} if

\[
\int (\bar{f} - f) \, d|\mu| = 0
\]

for every \( f \in C(K) \).

In the sequel \( \mathcal{F} \) and \( \mathcal{G} \) shall be the classes of all real valued functions on \( K \) which are pointwise limits of descending and ascending nets from \( \mathcal{H} \), respectively. Symbols such as \( \mathcal{F}_{\delta\sigma} \), \( \mathcal{G}_{\delta\sigma} \) etc., will be used in the customary meaning, \( \delta, \sigma \) denoting pointwise limits of descending and ascending sequences. The smallest class of functions containing \( \mathcal{F} \) and \( \mathcal{G} \) and being closed under pointwise limits of monotone sequences, will be denoted by \( \mathcal{A} \).

Clearly every function in \( \mathcal{F} \) is u.s.c. and affine, and every function in \( \mathcal{G} \) is l.s.c. and affine. The converse statements are also valid by virtue of the following:

\textbf{Proposition 1.} If \( f \) is an u.s.c. affine function on \( K \), then the set of all \( h \in \mathcal{H} \) such that \( f(x) < h(x) \) for all \( x \in K \), is directed downward. Consequently \( \mathcal{F} \) comprises all u.s.c. affine functions. Similarly \( \mathcal{G} \) comprises all l.s.c. affine functions.

\textbf{Proof.} Let \( h_i \in \mathcal{H} \) and \( f(x) < h_i(x) \) for all \( x \in K \) and \( i = 1, 2 \). Let \( \alpha, \beta \) be two real numbers bounding \( f, h_1, h_2 \) below and above, respectively, and define the following "ordinate sets" in \( E \times \mathbb{R} \)

\[
L = \{(x, \eta) \mid x \in K, \; \alpha \leq \eta \leq f(x)\},
\]

\[
U_i = \{(x, \eta) \mid x \in K, \; h_i(x) \leq \eta \leq \beta\}, \quad i = 1, 2 .
\]

Clearly \( L, U_1, U_2 \) are convex and compact, and by an elementary theorem, \( U = \text{conv}(U_1, U_2) \) is also compact.

The sets \( L \) and \( U \) are disjoint. In fact if \( (x, \eta) \in U \), then there is a convex combination

\[
x = \lambda y + (1 - \lambda)z, \quad 0 \leq \lambda \leq 1, \quad y, z \in K ,
\]

such that

\[
\eta \geq \lambda h_1(y) + (1 - \lambda)h_2(z) > \lambda f(y) + (1 - \lambda)f(z) = f(x) ,
\]

and hence \( (x, \eta) \notin L \).
By a well known separation property (based on the Hahn–Banach Theorem), the sets \( L \) and \( U \) may be separated by a hyperplane \( H \) in \( E \times R \). Now \( H \) is seen to be the graph of a continuous affine functional whose \( K \)-restriction has the desired property

\[
(1.4) \quad f(x) < h(x) < h_i(x)
\]

for all \( x \in K \) and \( i = 1, 2 \).

Now the last part of the proposition is an immediate consequence of (1.1).

Clearly \( \mathcal{A} \) is contained in the class \( \mathcal{B}_a \) of affine Borel functions, but the two classes are not identical in general. By an example of G. Choquet [3] (cf. also [6]) there exists an affine Borel function (of second Baire class) which does not enjoy the property (1.5) of our next proposition. The relationship between \( \mathcal{A} \) and \( \mathcal{B}_a \) is similar to the relationship between the monotone class generated by convex closed sets and the class of convex Borel sets. The latter two classes have been proved to coalesce in \( R^3 \) by V. Klee [5], but to the best of our knowledge the problem is open even for \( R^3 \).

**Proposition 2.** If \( \mu \) is a positive normalized measure with barycenter \( x \) and if \( f \) is a function of class \( \mathcal{A} \), then \( f \) is \( \mu \)-integrable and

\[
(1.5) \quad f(x) = \int f \, d\mu.
\]

**Proof.** Let \( \mathcal{K} \) be the class of all \( \mu \)-integrable functions of class \( \mathcal{A} \) for which (1.5) holds. If \( g \in \mathcal{G} \), then there is a net \( \{h_\alpha\} \) from \( \mathcal{K} \) such that \( h_\alpha \rightarrow g \). Now

\[
\sup_\alpha \int h_\alpha \, d\mu = \sup_\alpha h_\alpha(x) = g(x) < \infty,
\]

and by (1.2) \( g \) is integrable and

\[
\int g \, d\mu = \sup_\alpha \int h_\alpha \, d\mu = g(x).
\]

Hence \( \mathcal{G} \subset \mathcal{K} \). Similarly one may prove \( \mathcal{F} \subset \mathcal{K} \).

Next consider an increasing sequence \( \{f_n\} \) from \( \mathcal{K} \) which converges pointwise to a real valued function \( f \). Then

\[
\sup_n \int f_n \, d\mu = \sup_n f_n(x) = f(x) < \infty,
\]

and by the Monotone Convergence Theorem, \( f \) is integrable and

\[
\int f \, d\mu = \sup_n \int f_n \, d\mu = f(x).
\]
Hence \( f \in \mathcal{K} \). Similarly one may prove that \( \mathcal{K} \) is closed under pointwise limits of descending sequences. It follows that \( \mathcal{K} = \mathcal{A} \), and the proof is accomplished.

A non-zero signed boundary measure with total mass zero and resultant in the origin is said to be an affine dependence on \( \partial_{\varepsilon}K \), and \( K \) is said to be a simplex if there is no affine dependence on \( \partial_{\varepsilon}K \) (cf. [1]). By a theorem of G. Choquet and P. A. Meyer [2, p. 145], \( K \) is a simplex if and only if \( \tilde{f} \) is an u.s.c. affine function for every \( f \in \mathcal{I} \), or equivalently if and only if \( f \) is a l.s.c. affine function for every \( f \in -\mathcal{I} \). Hence it follows from Proposition 1, that \( K \) is a simplex if and only if \( \tilde{f} \in \mathcal{F} \) for every \( f \in \mathcal{I} \), or equivalently if and only if \( f \in \mathcal{F} \) for every \( f \in -\mathcal{I} \).

2. Hahn-decomposition by half-spaces.

We first prove that any two mutually singular boundary measures on a simplex can be "separated up to \( \varepsilon \)" by a function from \( \mathcal{K} \).

**Proposition 3.** If \( \mu \) and \( \nu \) are mutually singular, positive boundary measures on a simplex \( K \), then for every \( \varepsilon > 0 \) there exists an \( h \in \mathcal{K} \) such that \( 0 \leq h \leq 1 \) and

\[
\int h \, d\nu \leq \varepsilon, \quad \int (1-h) \, d\mu \leq \varepsilon.
\]

**Proof.** By the mutual singularity of \( \mu \) and \( \nu \) there exists a continuous function \( f \) on \( K \) such that \( 0 \leq f \leq 1 \) and

\[
\int f \, d\nu \leq \frac{1}{2} \varepsilon, \quad \int (1-f) \, d\mu \leq \frac{1}{2} \varepsilon.
\]

By (1.1) \( f \) is the supremum of the set of all \( g \in \mathcal{I} \) such that \( g(x) < f(x) \) for all \( x \in K \). This set is closed under finite suprema ("réticulé supérieur"). In particular it is directed upward, and by (1.2) it has a member \( g \) such that

\[
\int g \, d\mu \geq \int f \, d\mu - \frac{1}{2} \varepsilon.
\]

This inequality subsists with \( g^+ \) in the place of \( g \), and clearly \( g^+ \in \mathcal{I} \), \( 0 \leq g^+ \leq f \) and \( g^+(x) < 1 \) for all \( x \in K \). Hence by (2.2) and by the characteristic property (1.3) of boundary measures

\[
\int g^+ \, d\nu \leq \int f \, d\nu \leq \frac{1}{2} \varepsilon,
\]

and

\[
\int g^+ \, d\mu \geq \int f \, d\mu - \frac{1}{2} \varepsilon \geq \mu(K) - \varepsilon.
\]
Since $K$ is a simplex and $g^+ \in \mathcal{S}$, the function $g^+$ is u.s.c. and affine. By (1.1) $g^+$ is the infimum of the set of all $h \in \mathcal{H}$ such that $h(x) > g^+(x)$ for all $x \in K$. By Proposition 1 this set is directed downward and by (1.2) it has a member $h$ such that

$$\int h \, dv \leq \int g^+ \, dv + \frac{1}{4} \varepsilon.$$ 

We may assume $h \leq 1$ since $g^+(x) < 1$ for all $x \in K$. Hence $0 \leq h \leq 1$, and by (2.3), (2.4) and by use of (1.3) once more

$$\int h \, dv \leq \int g^+ \, dv + \frac{1}{4} \varepsilon \leq \varepsilon,$$

and

$$\int h \, d\mu \geq \int g^+ \, d\mu \geq \mu(K) - \varepsilon.$$

These relations complete the proof.

Proposition 4. Let $\mu$ and $\nu$ be mutually singular, positive boundary measures on a simplex $K$. For every $\varepsilon > 0$ there exists an (affine) function $g$ of class $\mathcal{A}_0$ such that $0 \leq g \leq 1$, and

$$\int g \, dv = 0, \quad \int (1 - g) \, d\mu < \varepsilon.$$

Moreover, there exists an (affine) function $f$ of class $\mathcal{A}_\infty$ such that $0 \leq f \leq 1$, and

$$\int f \, dv = \int (1 - f) \, d\mu = 0.$$

Proof. By Proposition 3 there exist functions $h_n \in \mathcal{H}$ such that $0 \leq h_n \leq 1$ and

$$\int h_n \, dv \leq 2^{-n}, \quad \int (1 - h_n) \, d\mu \leq 2^{-n},$$

for $n = 1, 2, \ldots$. Define

$$g_{n,p} = h_{n+1} \wedge \ldots \wedge h_{n+p}, \quad n, p = 1, 2, \ldots.$$ 

The functions $g_{n,p}$ are l.s.c. and affine since $K$ is a simplex. By Proposition 1,

$$g_{n,p} \in \mathcal{A}, \quad n, p = 1, 2, \ldots.$$ 

Now define

$$g_n = \inf_{p} g_{n,p}, \quad n = 1, 2, \ldots.$$ 

Clearly $g_n \in \mathcal{A}_0$, and
\[ \int g_n \, dv \leq \int h_{n+p} \, dv \leq 2^{-n-p}, \quad n, p = 1, 2, \ldots. \]

Hence
\[ (2.8) \quad \int g_n \, dv = 0, \quad n = 1, 2, \ldots. \]

By the characteristic property (1.3) of a boundary measure,
\[ \int (1 - g_{n,p}) \, d\mu = \int (1 - h_{n+1} \wedge \ldots \wedge h_{n+p}) \, d\mu \]
\[ \leq \sum_{k=n+1}^{n+p} \int (1 - h_k) \, d\mu \leq 2^{-n}(1 - 2^{-p}), \quad n, p = 1, 2, \ldots. \]

Hence by the Monotone Convergence Theorem
\[ (2.9) \quad \int (1 - g_n) \, d\mu = \sup_p \int (1 - g_{n,p}) \, d\mu \leq 2^{-n}, \quad n = 1, 2, \ldots. \]

By (2.8) and (2.9) the requirement (2.5) is satisfied with \( g = g_n \) when \( 2^{-n} \leq \varepsilon \).

Next define \( f = \sup_n g_n \). Clearly \( f \in \mathcal{G}_{\delta_0} \). By the Monotone Convergence Theorem and by (2.8)
\[ \int f \, dv = 0. \]

Clearly \( 1 - f \leq 1 - g_n \) for \( n = 1, 2, \ldots \). Hence by (2.9)
\[ \int (1 - f) \, d\mu = 0. \]

Thus, \( f \) has the desired property (2.6).

**Theorem 1.** A convex compact set \( K \) is a simplex if and only if every (signed) boundary measure \( \mu \) admits an affine function \( f \) of class \( \mathcal{A} \) such that
\[ (2.10) \quad \mu^-(\{x \mid f(x) \geq 0\}) = \mu^+\{x \mid f(x) \leq 0\}) = 0. \]

**Proof.** 1. Assume \( K \) to be a simplex. By Proposition 5 there exists an affine function \( g \) of class \( \mathcal{G}_{\delta_0} \) such that \( 0 \leq g \leq 1 \) and
\[ (2.11) \quad \int g \, d\mu^- = \int (1 - g) \, d\mu^+ = 0. \]

Let \( f = g - {\frac{1}{2}} \), and define \( A = \{x \mid f(x) \geq 0\}, \quad B = \{x \mid f(x) \leq 0\} \). Clearly \( {\frac{1}{2}} \chi_A \leq g, \quad {\frac{1}{2}} \chi_B \leq 1 - g \), and by (2.11)
\[ \mu^-(A) = \mu^+(B) = 0. \]

Thus \( f \in \mathcal{G}_{\delta_0} \subset \mathcal{A} \), and (2.10) is satisfied.
2. Assume $K$ to be a non-simplex. By the definition of a simplex there exists an affine dependence $\mu$ on $\partial K$. We assume the positive and negative parts of $\mu$ to be normalized, and we denote the common barycenter of $\mu^+$ and $\mu^-$ by $x$. Thus we have

\begin{align}
\mu^+(K) &= \mu^-(K) = 1 \\
\int t \, d\mu^+(t) &= \int t \, d\mu^-(t) = x
\end{align}

We claim that such a measure $\mu$ cannot admit any function $f$ of class $\mathcal{A}$ for which (2.10) is valid. In fact, assume $f \in \mathcal{A}$ and

\begin{equation}
\mu^-(A) = \mu^+(B) = 0 ,
\end{equation}

where $A = \{ x \mid f(x) \geq 0 \}$, $B = \{ x \mid f(x) \leq 0 \}$. By (2.12) and (2.14),

$$
\mu^+(\{ x \mid f(x) > 0 \}) = \mu^-(\{ x \mid f(x) < 0 \}) = 1 .
$$

Hence there is an $\alpha > 0$ such that

\begin{equation}
\mu^+(A_\alpha) \geq \frac{1}{2}, \quad \mu^-(B_\alpha) \geq \frac{1}{2} ,
\end{equation}

where $A_\alpha = \{ x \mid f(x) \geq \alpha \}$, $B_\alpha = \{ x \mid f(x) \leq -\alpha \}$. By virtue of (2.13), (2.15) and by Proposition 2

$$
\frac{1}{2} \alpha \leq \int_{A_\alpha} f \, d\mu^+ \leq \int f \, d\mu^+ = f(x) ,
$$

$$
-\frac{1}{2} \alpha \geq \int_{B_\alpha} f \, d\mu^- \geq \int f \, d\mu^- = f(x) .
$$

This contradiction completes the proof.

REFERENCES

6. R. Phelps, Lectures on Choquet’s theorem.

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