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# SPACES WITHOUT LARGE PROJECTIVE SUBSPACES

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## Introduction.

This paper is organized around a tabular array of 24 problems on non-projective or non-injective Boolean algebras or fields of sets—most of them now solved, many trivially. The 24 questions are these: for what cardinal numbers $m$ is the class of all (Boolean algebras/fields of sets) that have no infinite (complete/free/projective) (sub-/quotient) algebra or field closed under the formation of (free sums/direct products) of $m$-element families? 23 of them are answered, 18 completely and the other 5 depending more or less heavily on the continuum hypothesis or Ulam Measure Problem.

These 23 results involve approximately 3 ideas (beyond Stone's idea of duality between Boolean algebras and Boolean spaces, on which everything depends). The first is a recent result of Efimov and Katětov: every infinite dyadic bicompactum, hence every infinite injective Boolean space, contains a convergent sequence. The problems on projective algegras and fields then reduce to very simple forms, though one of them—whether a free sum of compact spaces containing no convergent sequences can contain a convergent sequence—seems to depend on the Ulam Problem. Second: for projective spaces a corresponding result is known; every infinite one admits a continuous mapping onto $\beta N$. But we need the stronger result that every infinite closed subspace of a projective space maps onto $\beta N$. With these additions to known results, we can essentially settle the 12 problems on algebras and 10 of the problems on fields of sets.

This work actually began with the question whether a direct product of pseudocompact proximity spaces must be pseudocompact. For zero-dimensional spaces, this is the dual form of the 23rd problem. Regardless of dimension, the theorem holds. The third idea is the characterization of pseudocompact spaces $X$ by extension of mappings: every $\delta$-continuous mapping of $X$ into a certain test space $T$ (goes into a compact subset of $T$; equivalently), can be extended over the $\delta$-compactification

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of $X$. For a large class of spaces $T'$, of which $T$ is one, that extension property is preserved by products.

The question remaining open: can a finite free sum of fields of sets have an infinite free subfield when its summands do not?

## 1. Injective spaces.

Let us repeat the central 24 problems and state the results. The terminology is standard for Boolean algebras but not for fields of sets. The generalized terminology and Stone duality will be explained after the statement of results. Perhaps it should be noted first that for both Boolean algebras and fields of sets, "complete" = "injective". (If it were not so, one would have 32 problems).

The questions: for what cardinal numbers $m$ is the class of all (Boolean algebras/fields of sets) that have no infinite (complete/free/projective) (sub-/quotient) algebra or field closed under the formation of (free sums/direct products) of $m$-element families?

We distinguish 12 *like* problems from 12 *cross* problems; the like problems are the 4 concerning products and completeness and the 8 concerning sums and freeness or projectiveness. The author expected to omit all details of these, merely observing that they all have positive solutions for finite cardinals, negative for infinite cardinals. Perhaps they do. Proofs will be indicated, except for a finite free sum of fields of sets without infinite free subfields.

Exactly 3 of the cross problems also have a positive solution for finite cardinals only. These are for products of algebras without infinite projective or free subalgebras, and products of fields without infinite projective subfields. As for free subfields, there is a countable product theorem; it fails for $c = 2^{\aleph_0}$. Using (in a trivial way) recent results of Cohen and Solovay on hypotheses contrary to the continuum hypothesis, we can show that the theorem is not provable for cardinals less than $c$. However, it might be settled without special hypotheses by being proved false for $\aleph_1$.

Six problems have positive solutions for enormous classes of cardinals, viz. those on infinite free or projective quotients and on infinite complete subobjects. For products of Boolean algebras without infinite free or projective quotients, we assume the number of factors is non-measurable; the other four work for all cardinals. The non-measurability may be removable. Bounded non-measurability of the factors can substitute for it, but this will not be proved. In summary, a supposed counterexample must be very complicated, and since Tarski can be cited [11] for the conjecture that the existence of measurable cardinals is inconsistent, the remaining problem does not seem worth more trouble.

The Boolean algebras which lack infinite complete quotients are closed under countable free sums, not provably closed for sums of larger families, not closed for $c$. For fields without infinite complete quotients, there is no sum theorem-assuming $c = \aleph_1$.

There are 22 positive results (one failing, one unsettled) and 18 negative results (6 problems turning out positive at least for non-measurable cardinals). 17 of each will be established, often with all details omitted, in this section of the paper. In fact, we dismiss 17 trivial counterexamples with the following words. Excepting the free sum of two fields without infinite complete quotient fields, the other examples are all easily found among finite Boolean algebras and fields of sets, free ones on $\aleph_0$ or $c$ generators, and the algebra or field of all subsets of a set of $\aleph_0$ or $c$ points. A careful reader should treat this as a set of exercises proposed after the definitions below.

We assume, without loss of generality, that all fields of sets are separated. Thus they are ordered triples $(B,X,i)$, where $B$ is a Boolean algebra, $X$ is a set, and $i$ is a homomorphism of $B$ into the algebra of all subsets of $X$ such that $b_1 \neq b_2$ in $B$ implies $i(b_1) \neq i(b_2)$, and $x_1 \neq x_2$ in $X$ implies that for some $b$ in $B$, $x_1 \in i(b)$ but $x_2 \notin i(b)$. (It follows that $i(1) = X$). One could define the dual *zero-dimensional proximity spaces* simply as ordered triples $(X,B,i)$ for which the same conditions hold. Using ordinary Stone duality, one could define the space instead as $(X,B,i^{\#})$, where $i^{\#}$ is a one-to-one function from $X$ onto a dense subset of the Stone space $\mathscr{S}(B)$. This presentation generalizes to give an arbitrary proximity space if we suppress $B$ and consider a pair $(X,e)$, $X$ being a set and $e$ a one-to-one dense embedding of $X$ in a compact space. We want both presentations [1].

In the $(B,X,i)$, $(X,B,i)$ notation, we are concerned with pairs $h: B_1 \to B_2$, $f: X_2 \to X_1$, where $h$ is a homomorphism, $f$ is a function, and $i_2 h = f^{-1} i_1$. Such a pair $(h,f)$, or $h$ alone, is called a *homomorphism* or *field mapping* from $(B_1,X_1,i_1)$ to $(B_2,X_2,i_2)$; the pair $(f,h)$, or $f$ alone, is a *$\delta$-continuous* mapping from $(X_2,B_2,i_2)$ to $(X_1,B_1,i_1)$. We shall now adopt the usual device of referring to an ordered triple by its first term (when the level of confusion permits). Given $h$ and $f$ as above, if $h$ is onto, then $f$ is one-to-one (but the converse is false); in this case $B_2$ is a *quotient field* of $B_1$ and $X_2$ is a *subspace* of $X_1$. If $f$ is onto, then $h$ is one-to-one (but

[1] For the general theory of proximity spaces, see [5]. Until Section 3 it suffices to think of them as spaces with distinguished compactifications; the $\delta$-continuous mappings are the restrictions of continuous mappings of the compactifications.

the converse is false); in this case $B_1$ is a *subfield* of $B_2$, and $X_1$ is a *δ-continuous image* of $X_2$. (In proximity spaces, as in topological spaces, one imposes stronger requirements for a quotient mapping. Probably similar distinctions will be wanted in fields of sets, but the notion of subfield is standard and the present notion of quotient seems right).

We define *free sum, direct product, retract, projective* and *injective* in the usual manner. Thus the direct product of fields $(B_\alpha, X_\alpha, i_\alpha)$ is the Cartesian product algebra represented on a disjointed union of the sets $X_\alpha$ by $i(\{b_\alpha\}) = \bigcup i_\alpha(b_\alpha)$. The direct product of spaces $X_\alpha$ is their Cartesian product set together with the smallest field of subsets making all coordinate projections δ-continuous. The dual constructs are free sums. Retracts of objects $K$ are the images of idempotent mappings $k: K \to K$. Projective objects, of the four varieties we will consider (Boolean algebras, fields of sets, and their duals), may be characterized simply as retracts of free objects; injective objects are the duals of projective objects. Free Boolean algebras are, of course, free sums of the 4-element algebra (free on one generator). Free compact spaces are free sums of one-point spaces. Similarly a *free field* of sets is a free sum of copies of the unique field representation of the 4-element algebra; the dual spaces are *Cantor spaces*. A *complete* field of sets is a completely isomorphic representation of a complete Boolean algebra $B$; thus $B$ must be atomic and represented on its set of atoms. The dual spaces are (equivalently) *free, projective*, or *δ-discrete*.

7 of the 8 positive results on projective algebras and fields reduce to trivialities once we show that every infinite injective Boolean (or zero-dimensional proximity) space has as a retract the Aleksandrov space $\alpha N$ consisting of a sequence converging to a point. Since $\alpha N$ is injective (a retract of a Cantor space), it suffices to show that it occurs as a subspace. If we call a subset $S$ of a topological space $X$ *sequentially closed* when there is no convergent sequence of points of $S$ converging to a limit in $X - S$, we have:

1.1. THEOREM [2]. *Every open, sequentially closed subspace of a Cantor space is closed.*

PROOF. Let $S$ be open and sequentially closed in a Cantor space $P = \Pi T_\alpha$, where the $T_\alpha$ are two-point spaces indexed by ordinal numbers $\alpha$. Consider the points of $P$ as functions, and let $f$ be a point of $P - S$.

[2] It is already known that this result is consistent with the usual axioms for set theory. In fact, the stronger result that two complementary sequentially closed subspaces must be closed follows from Mazur's work [7], provided there exist no uncountable weakly inaccessible cardinal numbers.

We shall construct a descending sequence of indices $\beta_1, \beta_2, \ldots$ such that each $g$ agreeing with $f$ up to $\beta_i$ and at $\beta_{i-1}, \ldots, \beta_1$ belongs to $P-S$; completing the construction, the descending sequence must be finite, and (as we arrange it) it cannot end until $\beta_n = 0$. This will show that $P-S$ is a neighborhood of each of its points, and $S$ is closed.

To avoid two cases, let there be a largest index $\beta_1$. Having $\beta_1, \ldots, \beta_i \neq 0$, consider the subspace $Q$ of $P$ consisting of functions agreeing with $f$ at these indices. Among the ordinals $\alpha \leqq \beta_i$, some of them have the property that there is $\gamma < \alpha$ such that $g \in Q$, $g = f$ on $[0, \gamma]$, and $g = f$ on $[\alpha, \beta_i]$ imply $g \in P-S$. This holds trivially if $\alpha = \gamma + 1$. It remains to prove that it holds for $\alpha = \beta_i$, so that the induction runs.

Suppose $\alpha$ is an $\omega$-limit of smaller ordinals $\gamma_j$, but no $\gamma_j$ does what we want. Thus there are functions $g_j$ in $Q$ agreeing with $f$ up to $\gamma_j$ and on $[\alpha, \beta_i]$, all belonging to $S$. Since $S$ is open, there are finite sets of indices $N_j$ such that every $h$ agreeing with $g_j$ on $N_j$ belongs to $S$. Let $N$ be the union of all $N_j$. Let $h_j$ agree with $g_j$ on $N$ and with $f$ everywhere else. The sequence $\{h_j\}$ lies in the compact metric space of all functions agreeing with $f$ except on $N$, and therefore it has a subsequence converging to a limit $h$. Now $h$ is in $Q$, since all $g_j$ and $h_j$ are in $Q$; similarly $h$ agrees with $f$ on $[\alpha, \beta_i]$. At an index $< \alpha$, almost all $g_j$ and $h_j$ agree with $f$, and so does $h$. Thus $h$ agrees with $f$ up to $\beta_i$ and at $\beta_{i-1}, \ldots, \beta_1$, whence $h \in P-S$, a contradiction.

Suppose the desired property established for all predecessors $\delta$ of $\alpha$, where $\alpha$ is a limit ordinal and not a countable limit. For each $\delta < \alpha$ let $c(\delta)$ be the smallest suitable $\gamma : g \in Q$, $g = f$ on $[0, \gamma]$, and $g = f$ on $[\delta, \beta_i]$ imply $g \in P-S$. Then $c(\delta) < \delta$, and if $\delta' > \delta$ then $c(\delta') \geqq c(\delta)$. But this implies that $c$ is bounded away from $\alpha$. (The known argument: otherwise one could find an ascending sequence $\delta_n$ interlaced with $c(\delta_{n+1}) > \delta_n$, and the common limit $\varepsilon$ could not have $c(\varepsilon) < \varepsilon$). Thus there is $\gamma < \alpha$ such that for all $\delta < \alpha$, if $g \in Q$, $g = f$ on $[0, \gamma]$, and $g = f$ on $[\delta, \beta_i]$, then $g \in P-S$. Finally, no $g$ in $Q \cap S$ agrees with $f$ even on $[0, \gamma]$ and $[\alpha, \beta_i]$; for $S$ is open, and every such $g$ is a limit of the functions just shown to be in $P-S$.

Continuous images of Cantor spaces are called *dyadic bicompacta*. Our injective spaces are instances; so is every compact group [6].

1.2. COROLLARY.[3] *In a dyadic bicompactum, every open, sequentially closed subspace is closed. Thus every non-isolated point is the limit of a*

[3] The second statement in 1.2 (which implies 1.1) is stated by B. Efimov [2]. He states that M. Katětov has proved the same result. Most of Efimov's proof is in his earlier note [1] (in Proposition D and the proof of Theorem 4). The argument proves more and is longer than the present one.

*convergent sequence of other points; and every infinite injective Boolean space contains $\alpha N$.*

PROOF. A routine check shows that the property in 1.1 is inherited by continuous images (closed continuous images, in general spaces). Thus if the complement of a point is not closed, it is not sequentially closed. Finally, an infinite compact space has a non-isolated point.

The injective Boolean spaces are seen to be sharply distinguished from the projective spaces, which contain no convergent sequences. (This last is known and will be strengthened below, 2.1). The fact that no infinite Boolean space is both projective and injective is stated without proof in [4], credited to D. Scott and H. Trotter. One of our 8 results requires exactly a proof that a free sum of Boolean spaces not containing $\alpha N$ never contains $\alpha N$. With a restriction, that will be in 2.3. The other seven present no difficulty.

Consider next four of the questions on complete algebras and fields, namely the ones about direct products. These are negative for infinite cardinals. The positive results concern embedding a free space on $\aleph_0$ points in a finite sum of spaces, and they are trivial.

There are 8 problems on Cantor spaces, which obviously reduce to the ordinary Cantor space $C$. One pair of them, for fields and algebras, are mutually equivalent; can $C$ be embedded in $X \times Y$ without being embeddable in $X$ or $Y$? It suffices to consider the images of $C$ in $X$ and $Y$, which are two compact metric spaces. But a compact metric space contains a Cantor set if and only if it is uncountable. Third question: can a free sum of proximity spaces $X_\alpha$ not containing $C$ contain $C$? No, for the sum is the union of open-closed sets $X_\alpha$, and a non-empty open-closed set of $C$ is homeomorphic with $C$. The corresponding result for compact spaces requires assurance that $C$ cannot be embedded in the set of points added to the union to compactify it; after 2.3, we will have that for non-measurable cardinal numbers. Next we consider a pair of questions having different answers in the compact and the non-compact case. For compact $X$ and $Y$, it is trivial that $X \cup Y$ cannot map onto $C$ without one of the images containing an open set and therefore admitting a mapping onto $C$. On the other hand, a countable sum $\beta N$ can map onto $C$. But for proximity spaces, sum is mere union.

1.3. PROPOSITION. *If the ordinary Cantor set $C$ is expressed as a countable union of subsets, at least one of them admits a uniformly continuous mapping onto $C$.*

PROOF. First suppose $C$ is a union of two sets, $A$, $B$, and $A$ does not map onto $C$. Then in every perfect subset $D$ of $C$ there is a perfect subset of $B$. To see this, observe that $D$ is homeomorphic with $C$ and thus with $C \times C$; moreover, a homeomorphism $h: D \to C \times C$ can be extended over $C$. Then the first coordinate of $h$ must not map $A \cap D$ onto $C$; thus $B \cap D$ contains one of the perfect sets $h^{-1}(p)$, $p \in C$.

It follows that a space mapping onto $C$ cannot be a finite union of subspaces not mapping onto $C$ (by considering images). Suppose $C$ is a countable union of sets $S_i$, with no $S_i$ mapping onto $C$. Let $D_1$ be a perfect set disjoint from $S_1$, and recursively let $D_{n+1}$ be a perfect subset of $D_n$ disjoint from $S_{n+1}$. By compactness, the $D$'s have a common point; but it can belong to no $S_i$, a contradiction.

From 1.3, a countable sum of proximity spaces not mapping onto $C$ cannot map onto $C$. If we assume the continuum hypothesis, this completely solves the sixth problem on Cantor spaces. Among the alternatives to the continuum hypothesis, P. Cohen and R. Solovay have shown independently that $c = \aleph_{\omega_1}$ is consistent (oral communication from Solovay); this implies readily that 1.3 is not true for all cardinals less than $c$.

Seventh (and last) result on $C$:

1.4. PROPOSITION. *The product of two compact spaces which admit no continuous mapping onto $C$ admits no continuous mapping onto $C$.*

PROOF. For metric spaces: if $X \times Y$ maps to $C$, the mapping factors across the component space $X' \times Y'$. Then that space is uncountable. Hence $X'$ or $Y'$ is an uncountable compact metric space, and contains a copy of $C$. Since the component spaces are Boolean, $C$ is a retract of $X'$ or $Y'$, and therefore an image of $X$ or $Y$.

For the general case, we need to factor across $X' \times Y'$ and then across $X'' \times Y''$, an intermediate metric product space. Probably the quickest way to see that this can be done is to use function spaces and the relation $C^{A \times B} = (C^A)^B$. This holds for compact spaces with the uniform topology on the function space, and that is a metric topology when the range $C$ is metric. Then $f': X' \times Y' \to C$ gives us a continous mapping of the compact space $X'$ into the metric space of mappings from $Y'$ to $C$; the image is a compact metric space $X''$, and $f'$ factors across $X'' \times Y'$. Another step finishes the proof.

The eighth question, whether 1.4 generalizes to proximity spaces and $\delta$-continuous mappings, can be reduced in the same way (precompactness replacing compactness). $X''$ and $Y''$ turn out to be embeddable in $C$. But the question remains open.

## 2. Projective compact spaces.

We have still four cross problems about free spaces on $\aleph_0$ points, and two promised results about free sums of compact spaces. Five of these matters are treated in this section.

The free sum of a family of compact spaces (if it exists) clearly must be a compactification of their discrete sum; indeed, one easily verifies that the Stone–Čech compactification of the discrete sum is the compact free sum (which therefore does exist). In particular, the free spaces are the Stone–Čech compactifications $\beta D$ of discrete spaces $D$, and the projective spaces are their retracts. We shall want another characterization of the projective spaces, as *extremally disconnected* compact spaces. The definition of an extremally disconnected space will not be used. (See [3]). But this: a subspace of a topological space $X$ is *normally embedded* if every bounded continuous real-valued function on the subspace extends continuously over $X$. $X$ is extremally disconnected if and only if $X$ is completely regular and every dense subspace is normally embedded [3, Exercise 6$M$]. If $X$ is extremally disconnected, so is $\beta X$ (same exercise). Hence to show that a compact space $Y$ is projective, it suffices to exhibit one dense subspace $X$ of $Y$ such that every dense subspace of $X$ is normally embedded in $Y$ (for $Y$ is then $\beta X$).

2.1. PROPOSITION. *Every $\sigma$-compact subspace of a free compact space is normally embedded.*

REMARK. One may note that this, if true, is true more generally for projective spaces. Indeed it is true still more generally, for subspaces of free spaces.

PROOF. If $A$ is a $\sigma$-compact subspace of the free space $\beta D$, we show that $D \cup A$ is a paracompact space (hence normal) by using the criterion that every open covering has a $\sigma$-locally finite open refinement [8]. Evidently there exist a countable open refinement on a neighborhood of $A$ and a discrete open refinement on the rest of $D \cup A$. Then since $A$ is closed in $D \cup A$, a bounded real-valued continuous function $f$ on $A$ has an extension $g$ over $D \cup A$. Let $h = g \mid D$; $h$ has an extension $i$ over $\beta D$. Since $i = g$ on $D$, $i = g = f$ on $A$.

2.2. PROPOSITION. *Every separable subspace of a free compact space is extremally disconnected. Hence every infinite closed subspace contains an infinite projective subspace.*

PROOF. If $C \subset \beta D$ has a countable dense set $B$, then $B$ is normally embedded and extremally disconnected. The closure $E$ of $B$ is extremally disconnected; and every dense subset of $C$ is dense in $E$, normally embedded in $E$, normally embedded in $C$. Finally, an infinite closed set $F$ in $\beta D$ contains a countably infinite set $A$ and its projective closure.

2.3. Proposition. *If $X$ is a locally compact $\sigma$-compact space, then every $\sigma$-compact subspace of $\beta X - X$ is normally embedded and every separable subspace is extremally disconnected. If $X$ is a discrete sum of non-measurably many locally compact $\sigma$-compact spaces, every infinite closed subspace of $\beta X - X$ contains an infinite projective subspace.*

Proof. First part: repeat the arguments of 2.1 and 2.2 except that every open covering of $X \cup A$ turns out to have a countable subcovering. For the second part, we have a mapping $f: X \to D$, where $D$ is a non-measurable discrete space and each set $f^{-1}(p)$ is locally compact $\sigma$-compact. Extend to $\beta f: \beta X \to \beta D$. If $F$ is an infinite closed subspace of $\beta X - X$, $\beta f(F)$ is finite or contains an infinite projective space $P$. In the second case, since $P$ is projective, $P$ can be embedded in $F$. In the first case some infinite closed subset $G$ of $F$ lies in an inverse set $(\beta f)^{-1}(p)$. If $p \in D$, then $G$ lies in $\beta[f^{-1}(p)] - f^{-1}(p)$ and contains an infinite projective subspace. The remaining possibility is $p \in \beta D - D$. The traces on $D$ of neighborhoods of $p$ determine a finitely additive non-atomic two-valued measure; this measure cannot be countably additive, so some countable family of neighborhoods $U_n$ of $p$ has intersection disjoint from $D$. Let $g_n$ be the characteristic function of $U_n$; $g = \Sigma 2^{-n}(1-g_n)$; $h$ the continuous extension of $g$ over $\beta D$. Let $Y$ be the set of all points $y$ of $\beta X$ such that $h(\beta f(y)) > 0$. Then $Y$ is an open $F_\sigma$ set of $\beta X$ which (i) contains $X$ and (ii) is disjoint from $(\beta f)^{-1}(p)$. From (i), $\beta X = \beta Y$. From (ii), $G \subset \beta Y - Y$; so the proof is complete.

We want a special property of the free space on $\aleph_0$ points, $\beta N$.

2.4. Theorem. *Every subspace $\beta N$ of a free compact space is a retract. Hence every infinite closed subspace retracts upon an infinite projective space.*

Proof. Given $\beta N \subset \beta D$, construct a sequence of pairwise disjoint neighborhoods $U_n$ of the isolated points $x_n$ of $\beta N$. Let

$$V_1 = D - \bigcup[U_n : n > 1], \qquad V_n = D \cap U_n$$

thereafter. Map $D$ to $N$ by putting $f(x) = x_n$ if $x \in V_n$. The mapping $f$ has a continuous extension $\beta f: \beta D \to \beta N$. Since $D$ is dense in $\beta D$, $\beta f(x_n) = x_n$, and $\beta f$ is a retraction.

Any infinite closed set $X$ in $\beta D$ contains a discrete subspace $N$ and its closure $\beta N$, and $X \subset \beta D$ retracts upon $\beta N$. This completes the proof.

One naturally asks whether every continuous mapping of a closed subspace of $\beta D$ into $\beta N$ can be extended over $\beta D$. This is entirely contrary to "known" results about these spaces; but those results depend on the continuum hypothesis, and without it, it is not clear how to construct mappings except for the extensible mappings. If the continuum hypothesis is assumed, there are $2^c$ autohomeomorphisms of $\beta N - N$ [10], and it is not hard to see that only $c$ of them can be extended to continuous mappings $\beta N \to \beta N$.

The author is indebted to M. Henriksen for asking whether in Theorem 2.4, $\beta N$ can be replaced by an arbitrary projective space. It is a striking idea, and entirely open.

2.5. THEOREM. *Let $\{X_i : i \in I\}$ be a family of compact spaces none of which has an infinite projective quotient space. Then their product has no infinite projective quotient space.*

PROOF. Assume the contrary. Then the product $P$ admits a continuous mapping $f$ onto $\beta N$; but no factor $X_i$ can be mapped upon an infinite subset of $\beta N$. Consider the inverse sets $f^{-1}(p_n)$ of the isolated points of $\beta N$. Each is open and compact, hence restricted in only finitely many coordinate indices. Uniting these index sets, we discover a countable partial product, every cross-section over which maps onto $\beta N$. Then we may assume $I$ is the set of positive integers. (If it was finite, fill out with one-point factor spaces). Now we shall get a contradiction between $\aleph_0^{\aleph_0} = c$ and the fact that $\beta N$ has more than $c$ points [3, p. 130].

Choose points $x^n = (x_i^n) \in f^{-1}(p_n)$. Let $L_i$ be the set of all $x_i^n$. The product of all $L_i$ has $c$ points, and its closure $L$ maps onto $\beta N$. For each $p \in \beta N$, consider various points $\lambda$ of finite products $\Pi_{i \leq k} L_i$, and the cylinders

$$C_\lambda = \{x \in L: \text{ for } i \leq k,\ x_i = \lambda_i\}.$$

Suppose $C_\lambda$ contains a point $x$ of $f^{-1}(p)$. The set

$$S_x = \{y \in P: \text{ for } i \neq k+1,\ y_i = x_i\}$$

is homeomorphic with $X_{k+1}$, so $f(S_x)$ is finite. Then $f^{-1}(p) \cap S_x$ is relatively open; since $x \in L$, $x_{k+1} \in \bar{L}_{k+1}$, there is an $l \in L_{k+1}$ such that the cylinder over $\lambda' = (\lambda_1, \ldots, \lambda_k, l)$ meets $f^{-1}(p)$. As this holds for $k = 0$, we conclude that there is an infinite sequence $(\lambda_i)$, the cylinder over every initial segment of which meets $f^{-1}(p)$. Then $(\lambda_i)$ itself is in $f^{-1}(p)$, and the contradiction is established.

2.6. THEOREM. *Let $\{X_i\}$ be a countable family of Hausdorff spaces none of which has an infinite projective compact subspace. Then their product has no infinite projective compact subspace.*

PROOF. Case 1: a product $X \times Y$. Given an embedding $e: \beta N \to X \times Y$, we shall find an infinite closed subset of $\beta N$ on which one of the coordinate functions $e_1: \beta N \to X$, $e_2: \beta N \to Y$, is one-to-one. This is trivial unless each $e_i$ is finite-to-one; for when $e_1$ is constant on a set $S$, $e_2|S$ is one-to-one. Next, the sets $e_1^{-1}(x)$ must be of bounded finite size. Otherwise there would be a sequence of increasingly large sets $e_1^{-1}(x_j)$, whose union is normally embedded in $\beta N$ (by 2.1). As $\{x_j\}$ lies in a regular space $e_1(\beta N)$, it has an infinite discrete subset $D$. It follows that $e_1^{-1}(D)$ is also discrete. For any limit point $x$ of $D$, there are infinitely many disjoint subsets $S_k$ of $e_1^{-1}(D)$ such that $x \in e_1(S_k^-)$; thus $e_1^{-1}(x)$ is infinite.

Let $m$ be the largest integer such that there exist $\aleph_0$ different $m$-point sets $e_1^{-1}(x_j)$. Their union $V$ is normally embedded in $\beta N$ and can be written as a union of disjoint sets $V_1, \ldots, V_m$ each of which is mapped one-to-one by $e_1$. Each point of $V^-$ belongs to an inverse set containing at least $m$ points, and with finitely many exceptions, each of these sets consists of $m$ points, one from each $V_k^-$. We can remove neighborhoods of the exceptional points, leaving an infinite closed subset $W_k$ of each $V_k^-$. Then $e_1$ embeds $W_1$ in $X$.

The theorem follows for finite products.

Case 2: a countably infinite product. We have coordinate functions $e_i$, no finite set of which suffices to separate points on $\beta N$. Then for each $n$ there exist points $p_n, q_n$ of $\beta N$ such that $e_i(p_n) = e_i(q_n)$ for $i \leqq n$. Since countable sets are normally embedded, we may suppose that all the points $p_n, q_m$ are distinct. But for any limit point $p^*$ of the $p$'s there is a limit point $q^*$ of the $q$'s such that $e_i(p^*) = e_i(q^*)$ for all $i$; and since countable sets are normally embedded, $q^* \neq p^*$.

Briefly, 2.6 says that if $\beta N$ is embedded in $X \times Y$ then an infinite closed subset must be embedded in one factor. Since $\beta N$ has isolated points, there must also be a non-empty open subset embedded in one factor. It seems unlikely that any compact space without isolated points can have that property. We can show, assuming the continuum hypothesis, that $\beta N - N$ does not have it.

2.7. EXAMPLE. *Assuming $2^{\aleph_0} = \aleph_1$, there exist two quotient mappings $f: \beta N \to X$, $g: \beta N \to Y$, such that neither is one-to-one on any infinite open-closed subset but $x \to (f(x), g(x))$ is one-to-one, and $X$ and $Y$ are Boolean spaces. Thus $f(N)$ and $g(N)$ support fields of sets whose free sum has an infinite complete quotient field although neither summand has.*

PROOF. Let $N^*$ denote $\beta N - N$. We shall construct $p: N^* \to P$, $q: N^* \to Q$, neither one-to-one on any non-empty open set but together giving an embedding of $N^*$ in $P \times Q$. Then $X$ will be the quotient of $\beta N$ by the decomposition into sets $p^{-1}(t)$ and single points of $N$, and $Y$ is constructed similarly from $q$. Evidently $X$ and $Y$ will be Boolean if $P$ and $Q$ are. The zero-dimensional proximity spaces $f(N) \subset X$, $g(N) \subset Y$ then have no infinite subspace $N'$ whose $\delta$-compactification is $\beta N'$, thus no infinite projective subspace; but in $f(N) \times g(N)$ the diagonal will be projective, completing the proof.

The space $N^*$ has just $c$ open-closed sets, the derived sets of subsets of $N$ in $\beta N$. By hypothesis, then, the non-empty open-closed sets can be arranged in a list $\{U_\alpha\}$ indexed by the countable ordinals. We shall refer to these briefly as *listed* sets. Note that every closed $G_\delta$ set is the closure of its interior [10].

We generate labels $V_\alpha, S_\beta^\alpha, T_\beta^\alpha, \alpha > \beta$, for certain listed sets, and $X_\alpha$ for certain closed $G_\delta$ sets, as follows. $p_\alpha$ will be the characteristic function of $V_\alpha$. Let $V_0 = U_0$. At an ordinal $\alpha = \beta + 1$, let $x_\beta$ be a point of $V_\beta$. Let $X_\beta$ be the set of all points of $N^*$ not separated from $x_\beta$ by any of the functions $p_\gamma$, $\gamma \leqq \beta$. Let $S_\beta^\alpha$ and $T_\beta^\alpha$ be disjoint listed subsets of (the interior of) $X_\beta$. For $\gamma < \beta$, let $S_\gamma^\alpha = S_\gamma^\beta - V_\beta$, $T_\gamma^\alpha = T_\gamma^\beta - V_\beta$. Let $V_\alpha$ be a listed subset of $U_\alpha$ not containing any $S_\gamma^\alpha$ or $T_\gamma^\alpha$, $\gamma \leqq \beta$; such a set must exist, for $U_\alpha$ (being homeomorphic with $N^*$) has $c$ disjoint listed subsets [10]. Since the $V$'s are chosen in this way, the sets $S_\gamma^\alpha$, $T_\gamma^\alpha$ are indeed non-empty and therefore are listed. At a limit ordinal $\alpha$, let $S_\gamma^\alpha$, $T_\gamma^\alpha$ be listed subsets of the $G_\delta$ sets $\bigcap_{\beta<\alpha} S_\gamma^\beta$, $\bigcap_{\beta<\alpha} T_\gamma^\beta$ respectively, and choose $V_\alpha$ as before.

The functions $p_\alpha$ are the coordinates of a mapping $p'$ into a Cantor space; let $P = p'(N^*)$, and define $p: N^* \to P$ by $p(x) = p'(x)$. Then $P$ is Boolean. Observe that $p$ is not one-to-one on any non-empty open set $U$. For $U$ contains a listed set $U_\beta$. The coordinate functions $p_\gamma$ $(\gamma \leqq \beta)$ are constant on $S_\beta^{\beta+1} \cup T_\beta^{\beta+1}$; and for $\alpha > \beta$, the function $p_\alpha$ is constant on $S_\beta^{\alpha+1} \cup T_\beta^{\alpha+1}$. Thus the disjoint sets $\bigcap_{\alpha>\beta} S_\beta^\alpha$ and $\bigcap_{\alpha>\beta} T_\beta^\alpha$ are not separated by $p$.

On the other hand, each non-empty open set $U$ of $N^*$ contains a listed set $U_\alpha$ and an open inverse set $V_\alpha = p_\alpha^{-1}(1)$. Thus for each listed $U_\beta$ the obstruction set

$$A_\beta = p^{-1}[p(U_\beta) \cap p(N^* - U_\beta)]$$

is nowhere dense. So $q$ can be constructed as follows.

We generate disjoint families $\mathscr{L}_\alpha$ of listed sets, mappings $q_\alpha$, closed nowhere dense sets $R_\alpha$, closed $G_\delta$ sets $W_\alpha$, and listed sets $Y_\beta^\alpha$, $Z_\beta^\alpha$ $(\alpha \geqq \beta)$.

Here $\mathscr{L}_\alpha$ will always have dense union; $R_\alpha$ will be the remainder of $N^*$; and $q_\alpha$ will be a quotient mapping corresponding to the decomposition into the elements of $\mathscr{L}_\alpha$ and the single points of $R_\alpha$. (Thus the quotient space is Boolean). To begin, let $\mathscr{L}_0$ be a maximal family of listed sets disjoint from each other and from $A_0$. At each ordinal $\alpha=\beta+1$, select a point $w_\beta$ in $U_\beta$ but not in the first category set $\bigcup_{\gamma\leqq\beta}R_\gamma$. Let $W_\beta$ be the set of all points of $U_\beta$ not separated from $w_\beta$ by any of the functions $q_\gamma$, $\gamma\leqq\beta$. Let $Y_\beta{}^\beta$, $Z_\beta{}^\beta$ be disjoint listed sets interior to $W_\beta$. Let $K_0{}^\alpha$ be a listed set disjoint from $A_\alpha$, meeting both $Y_0{}^\beta$ and $Z_0{}^\beta$, but not containing any $Y_\gamma{}^\beta$ or $Z_\gamma{}^\beta$. (Such a set exists since there are $c$ disjoint listed sets meeting $Y_0{}^\beta$ and $Z_0{}^\beta$ but not $A_\alpha$). Recursively select $K_\gamma{}^\alpha$ disjoint from $A_\alpha$ and the preceding sets $K_\delta{}^\alpha$, meeting $Y_\gamma{}^\beta$ and $Z_\gamma{}^\beta$, but such that its union with its predecessors $K_\delta{}^\alpha$ still does not contain any $Y_\varepsilon{}^\beta$ or $Z_\varepsilon{}^\beta$. Extend the resulting countable family $\{K_\gamma{}^\alpha\}$, $\gamma<\alpha$, to a maximal family $\mathscr{L}_\alpha$ of listed sets disjoint from each other and from $A_\alpha$. Also

$$(*)\qquad Y_\gamma{}^\alpha = Y_\gamma{}^\beta \cap K_\gamma{}^\alpha, \qquad Z_\gamma{}^\alpha = Z_\gamma{}^\beta \cap K_\gamma{}^\alpha .$$

This carries the construction to the next limit ordinal $\alpha$, where we choose listed sets $Y_\gamma{}^\alpha$, $Z_\gamma{}^\alpha$ in $\bigcap_{\beta<\alpha}Y_\gamma{}^\beta$ and $\bigcap_{\beta<\alpha}Z_\gamma{}^\beta$ respectively. $K_0{}^\alpha$ and the rest of $\mathscr{L}_\alpha$ are constructed in the same way except that the index $\alpha$ replaces $\beta$ throughout. (*) is omitted.

The mapping $q$ is defined by its coordinates $q_\alpha$, the space $Q$ being the subspace $q(N^*)$ of the appropriate product space. Like $p$, $q$ is not one-to-one on any open set, because each listed set $U_\beta$ contains the nonempty disjoint sets $\bigcap_{\alpha>\beta}Y_\beta{}^\alpha$ and $\bigcap_{\alpha>\beta}Z_\beta{}^\alpha$, which no $q_\alpha$ separates. However, any distinct points $x\in U_\alpha$, $y\notin U_\alpha$ are either separated by $p$ or in $A_\alpha$ and separated by $q_\alpha$. The example is therefore established.

## 3. Pseudocompact proximity spaces.

We note again, this time more seriously, that basic facts on proximity spaces and uniform spaces can be found in [5]. It is not an encyclopedic reference; we must add, e.g., that the completion of a product of uniform spaces is the product of the completions of the factors [3, Exercise 15$M$]. A locally complete space is open in its completion; we omit proof.

3.1. Theorem. *For any locally complete uniform space $K$, the class of all uniform spaces $X$ such that every uniformly continuous mapping of $X$ into $K$ has a uniformly continuous extension over the completion of $X$ (into $K$) is closed under formation of direct products.*

Proof. Let $P$ be a product of such spaces $X_\alpha$, $Q$ the product of their completions $Y_\alpha$. Every map $f:P\to K$ can be extended, uniquely, to a map $g$ of (the completion) $Q$ into the completion $K^-$ of $K$. We wish to prove $g(Q) \subset K$, given that, for any copy of any $Y_\alpha$ (in $Q$), either $g(Y_\alpha) \subset K$ or $g(X_\alpha) \not\subset K$. Suppose some point $q^0 = (q_\alpha{}^0)$ of $Q$ maps to $z \in K^- - K$. In the closed set $g^{-1}(K^- - K)$ we construct a transfinite sequence of points $q^\alpha$ as follows. Well-order the indices $\alpha$. Having $q^\beta = (q_\alpha{}^\beta)$, consider the set

$$\{q \in Q: \text{ for } \alpha \neq \beta,\ q_\alpha = q_\alpha{}^\beta\}.$$

This is a section over $Y_\beta$, and $g$ does not map it into $K$. Hence it includes a point $q^{\beta+1}$ such that $q_\beta{}^{\beta+1} \in X_\beta$, $g(q^{\beta+1}) \notin K$. At a limit ordinal $\alpha$, the preceding $q^\beta$ form a convergent sequence; let $q^\alpha$ be their limit. The $\alpha$-th coordinate of $q^\beta$ is $q_\alpha{}^{\alpha+1} \in X_\alpha$ for all $\beta > \alpha$, so we arrive finally at a point $p$ of $P$; but $g(p) \in K^- - K$, a contradiction.

3.2. Corollary. *The class of fields of sets having no infinite complete subfield is closed under formation of free sums.*

Proof. Translate the proposition first by Stone duality and then by embedding in the category of uniform spaces. We arrive at the class of precompact uniform spaces $X$ such that $\delta dX = 0$ and every uniformly continuous image of $X$ in $N \subset \beta N$ is finite. A product $P$ of these is precompact zero-dimensional and, by 3.1, every mapping into $N \subset \beta N$ extends over the completion $P^-$. Since $P^-$ is compact, the image is a compact subspace of $N$; so it is finite.

We note three additions to 3.1. First, even if $K$ is not locally complete, the proof works for finite products. Second, in the precompact case (all $X^\alpha$ precompact, $K$ precompact and locally compact), one has closure under arbitrary products for the spaces $X$ such that every uniformly continuous image of $X$ in $K$ is compact; for this class is the intersection of the class described in 3.1 with reference to $K$ itself and all the classes similarly determined by subspaces $K -$ {point}. Third, a direct extension:

3.1*. Theorem. *Theorem* 3.1 *remains true if $K$ is merely a cone whose base is locally complete, or more generally, if the set of points at which $K$ is not locally complete is a complete subspace of $K$.*

We outline the proof, which depends on the use of single points rather like the preceding remark. If $f: P \to K$ fails to extend over $P^-$, this means the extension $g: P^- \to K^-$ assumes a value $z$ in $K^- - K$. If $S$ is a complete subspace of $K$, $z$ has a neighborhood $U$ far from $S$; if $S$ is the set of all points at which $K$ is not locally complete, then (taking $U$

closed or open) $K \cap U$ is locally complete. By routine arguments one gets a uniform covering $\{V_\beta\}$ of $P$ such that for some $\beta$, $f(V_\beta) \subset U$ and $z \in f(V_\beta)^-$. One can have $V_\beta$ a rectangle, and then the proof of 3.1 shows that on some section across $V_\beta$, $f$ cannot be extended over the completion with values in $K \cap U$. If we took $U$ closed, prolongation of that section gives a factor of $P$ on which $f$ fails to extend over the completion.

Doubtless 3.1 admits other extensions. The purpose of the extension to cones is to include the class of *pseudocompact* proximity spaces: the spaces admitting no non-precompact compatible uniformity, or equivalently [5, Exercise II.11], having a unique compatible uniformity. It remains to show that this purpose has been achieved. (We remark, but do not prove, that it cannot be done without cones; no locally compact space suffices to test for pseudocompactness).

We need the result [5, Exercise II.12] that if $Y$ and $Y'$ are $\delta$-isomorphic uniform spaces and $X$ is precompact, then $X \times Y$ and $X \times Y'$ are $\delta$-isomorphic. It implies

3.3. Proposition. *A $\delta$-continuous image of a pseudocompact proximity space is pseudocompact.*

Proof. Let $f: X \to Y$ be $\delta$-continuous onto. Transferred into uniform spaces, $X$ and $Y$ are precompact and $f$ is uniformly continuous. The graph $X'$ of $f$ in $X \times Y$ is uniformly isomorphic with $X$, since projection $(x,f(x)) \to x$ and the mapping $x \to (x,f(x))$ are both uniformly continuous. If $Y$ is not pseudocompact, there is a $\delta$-isomorphic uniformity making $Y$ into a non-precompact space $Y'$. $X \times Y'$ is $\delta$-isomorphic with $X \times Y$, so the point set $X'$ becomes a subspace $X''$ that is still $\delta$-isomorphic with $X$. But $X''$ is not precompact; a cylindrical covering of $X \times Y'$ over a uniform covering of $Y'$ that has no finite subcovering can have no finite subfamily covering $X''$.

We shall let $fN$ denote either the countably infinite free proximity space or the corresponding precompact uniform space. With corresponding ambiguity, $T$ denotes the cone over $fN$.

3.4. Proposition. *A proximity space is pseudocompact if and only if it has no infinite free subspace that is a retract of a $\delta$-neighborhood of itself.*

3.5. Proposition. *A proximity space $X$ is pseudocompact if and only if every $\delta$-continuous mapping of $X$ into $T$ extends over the $\delta$-compactification of $X$. If $X$ has $\delta$-dimension* 0, *the same holds with $T$ replaced by $fN$.*

PROOF OF 3.4. A non-precompact uniform space has an infinite uniformly discrete subspace $N$, and $N$ is a uniform neighborhood retract wherever it is embedded [5, p. 81]. Passing to proximity spaces, we have a $\delta$-neighborhood retracting upon $fN$.

Conversely, suppose a $\delta$-neighborhood $U$ retracts upon $fN \subset X$. Let $V$ be a smaller $\delta$-neighborhood far from $X - U$; let $V_n$ be the inverse image of $n \in N$. Pass to precompact uniform spaces $fN \subset gX$ and define a new uniformity $h$ on $X$ as follows. For each uniform covering $\{W_i\} \in g$, form the covering consisting of all $W_i$ that meet $X - V$ and all sets $V_n \cap W_i$. Evidently these coverings satisfy the necessary conditions [5, p. 5] to form a basis for a uniformity $h$. $hX$ is not precompact, for its subspace $N$ is uniformly discrete. The identity mapping from $hX$ to $gX$ is $\delta$-continuous since $g \subset h$. For its inverse, consider any two sets $A$, $B$, near in $gX$. At least the uniform neighborhoods of subsets of $X - V$ are the same in $gX$ and in $hX$; so if $A - V$ is near $B$ or $B - V$ is near $A$, then $A$ and $B$ are near in $hX$. If neither of these relations holds, then $A \cap V$ is near $B \cap V$ in $gX$. But on $V$, the uniformity $h$ is obtained from $g$ in the same way as in the proof of 3.3; hence $hX$ and $gX$ are $\delta$-isomorphic.

PROOF OF 3.5. Suppose $f: X \to T$ is $\delta$-continuous and not extensible over the $\delta$-compactification of $X$. Then in the $\delta$-compactification of $T, f(X)$ has a limit point not in $T$; from the form of $T$ it is clear that $f(X)$ is not pseudocompact. Hence $X$ is not pseudocompact. Conversely, if $X$ is not pseudocompact, 3.4 gives us a retraction of a $\delta$-neighborhood $V$ of $fN \subset X$ upon $fN$. Combining this with a mapping into $[0,1]$ that is 0 on $N$ and 1 outside $V$ [5, p. 7], we get a mapping into $T$ that identifies $fN$ in $X$ with the base of the cone, and cannot extend over a compactification of $X$. In the zero-dimensional case, the same arguments apply with $fN$ in place of $T$ except that we choose $V$ to be far from $X - V$ and map $X - V$ to $fN$ in any manner.

3.6. THEOREM. *Every product of pseudocompact proximity spaces is pseudocompact.*

This is immediate from 3.1* and 3.5. In view of the second part of 3.5, this result generalizes 3.2.

Theorem 3.6, for finite products, has been announced by V. Poljakov [9].

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