A GENERALIZATION OF A THEOREM
OF SYLVESTER ON THE LINES DETERMINED
BY A FINITE POINT SET

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In 1893 Sylvester stated without proof: Given any finite set of non-
collinear points in the real projective plane, there exists at least one line
which contains exactly two of the given points.

This theorem remained unproved for nearly 40 years. In 1933 T. Gal-
lai (Grünwald) gave a proof, and later on several other proofs have been
found (for references cf. [3, p. 451] and [2, p. 65]). Here we mention
R. Steinberg’s (1944, cf. [1, p. 30]) because related ideas are used in the
present paper.

In general it is not true that, given a finite point set in a projective
space of dimension \( d > 2 \) which is not contained in a hyperplane, there
exists a hyperplane containing exactly \( d \) of the given points. A counter-
example in 3-space is a set of 6 points, 3 on each of two skew lines. An-
other one is the Desargues configuration in 3-space [3, p. 452].

The following generalization of Sylvester’s theorem has been proved
by Th. Motzkin [3] for \( d = 3 \):

Given a finite point set \( \Gamma_0 \) in the \( d \)-dimensional real projective space which
is not contained in a hyperplane. Then among the hyperplanes determined by
points of \( \Gamma_0 \) there is at least one with the property that the points of \( \Gamma_0 \) which
it contains, with the exception of precisely one of them, lie in a \( (d-2) \)-
dimensional projective subspace.

It is the aim of this paper to give a proof of this theorem for arbitrary
dimension \( d \).

Let \( \Gamma_p, p = 0, 1, \ldots, d \), denote the set of the projective subspaces of
dimension \( p \) which are spanned by the points of the given set \( \Gamma_0 \). Since
\( \Gamma_0 \) spans the whole \( d \)-dimensional space \( P_d \) considered, none of these sets
is empty, and \( \Gamma_d \) consists of \( P_d \) only. The elements of \( \Gamma_p \) will be denoted
\( A_p, \ldots, P_p, \ldots \). The union

Received December 3, 1963.
\[ \Gamma = \bigcup_{p=0}^{d} \Gamma_p \]

will be called the configuration \( \Gamma \).

A subspace \( A_p \in \Gamma_p \) will be called elementary if it contains exactly \( p+1 \) points of \( \Gamma_0 \). If \( \Gamma_0 \) consists of \( d+1 \) points, then \( \Gamma \) consists of the subspaces spanned by the vertices of a simplex, and thus all elements of \( \Gamma \) are elementary. In this case the configuration \( \Gamma \) will be called elementary.

If a subspace \( A_p \) is spanned by a point \( B_0 \) and a subspace \( C_{p-1} \), we shall write

\[ A_p = B_0 C_{p-1} . \]

Obviously, \( B_0 \in \Gamma_0 \) and \( C_{p-1} \in \Gamma_{p-1} \) imply \( A_p \in \Gamma_p \). If in this case \( B_0 \) is the only point of \( \Gamma_0 \cap A_p \) outside \( C_{p-1} \), then the subspace \( A_p \) is called ordinary. (This makes also sense for \( p = 0 \), if, as usual, the \((-1)\)-dimensional projective space is defined to be the empty set.) It is clear that every elementary subspace is ordinary. In this terminology the theorem above states the existence of an ordinary hyperplane.

Every \( A_p \in \Gamma, p > 0, \) is divided into polyhedral domains by the subspaces \( A_{p-1} \in \Gamma_{p-1} \) contained in \( A_p \). The closures of these domains will be called the \( p\)-dimensional cells of the configuration \( \Gamma \). Each cell \( \delta_p \) in \( A_p \) is obviously convex in the sense that, for any two points of it, that segment joining these points which does not intersect an \( A_{p-1} \subset A_p \) belongs entirely to \( \delta_p \).

We shall prove the following theorem which is slightly stronger than the statement above:

**Theorem.** Suppose that the configuration \( \Gamma \) in the real projective space of dimension \( d \) is not elementary, and let \( \delta_d \) be a \( d \)-dimensional cell of \( \Gamma \). Then there exists an ordinary hyperplane \( A_{d-1} = B_0 C_{d-2} \), where \( B_0 \in \Gamma_0 \) and

\[ A_{d-1} \cap \Gamma_0 \setminus \{B_0\} \subset C_{d-2} \subset \Gamma_{d-2} , \]

such that

\[ A_{d-1} \cap \delta_d \subset C_{d-2} . \]

We start by proving two lemmas.

**Lemma 1.** Let \( \sigma_d, d > 0, \) denote a closed \( d \)-dimensional simplex whose vertices belong to \( \Gamma_0 \), and suppose that the point \( A_0 \in \Gamma_0 \) is not contained in \( \sigma_d \). Then there is a \((d-2)\)-dimensional face \( \sigma_{d-2} \) of \( \sigma_d \) such that the hyperplane \( B_{d-1} \in \Gamma_{d-1} \) spanned by \( A_0 \) and \( \sigma_{d-2} \) satisfies

\[ B_{d-1} \cap \sigma_d = \sigma_{d-2} . \]
Proof. Consider the hyperplanes which contain the \((d - 1)\)-dimensional faces of \(\sigma_d\). Every pair of distinct such hyperplanes divides the space \(P_d\) into two "wedges". The closure of one of these contains \(\sigma_d\). Since \(\sigma_d\) is the intersection of such closed wedges, there must be at least one which does not contain \(A_0\). The hyperplane spanned by \(A_0\) and the intersection of the hyperplanes bounding such a wedge satisfies the requirement of the lemma.

Lemma 2. Let \(A_0\) be a point of \(\Gamma_0\), and let \(C_{d-1} \in \Gamma_{d-1}\) be a hyperplane which does not contain \(A_0\). Let further \(\delta_{d-1} \subset C_{d-1}\) be a \((d - 1)\)-dimensional cell of \(\Gamma\). If \(P_0\) is a point such that the line \(A_0P_0\) does not meet \(\delta_{d-1}\) and \(Q_0\) is an interior point of \(\delta_{d-1}\), then each of the two segments \(P_0Q_0\) intersects at least one of the hyperplanes, belonging to \(\Gamma_{d-1}\), which are spanned by \(A_0\) and the \((d - 2)\)-dimensional faces of \(\delta_{d-1}\).

Proof. The union of the lines joining \(A_0\) with the points of \(\delta_{d-1}\) is a polyhedral convex cone. Since \(P_0\) is an exterior point and \(Q_0\) an interior point of this cone, each of the segments \(P_0Q_0\) intersects its boundary. The statement now follows from the fact that every boundary point of the cone is contained in at least one of the hyperplanes spanned by \(A_0\) and the \((d - 2)\)-dimensional faces of \(\delta_{d-1}\).

Proof of the Theorem. We proceed by induction on the dimension of the space. For \(d = 1\) the Theorem is obvious. Let \(d > 1\) be given. We assume the Theorem to be true for spaces of dimension \(d - 1\).

Let a \(d\)-dimensional cell \(\delta_d\) of \(\Gamma\) be given. Obviously, \(\delta_d\) is contained in some closed simplex with vertices belonging to \(\Gamma_0\). If this simplex contains a point of \(\Gamma_0\) different from its vertices, the hyperplanes spanned by this point and the vertices divide the simplex into smaller simplexes one of which contains \(\delta_d\). If this simplex contains a point of \(\Gamma_0\) different from its vertices, the procedure can be repeated. After finitely many steps a simplex \(\sigma_d\) is obtained which contains \(\delta_d\) but no point of \(\Gamma_0\) other than its vertices.

Since \(\Gamma\) is not elementary, there is a point of \(\Gamma_0\) outside \(\sigma_d\). By Lemma 1, there exists a hyperplane \(B_{d-1} \in \Gamma_{d-1}\) through this point for which

\[
B_{d-1} \cap \delta_d \subset B_{d-1} \cap \sigma_d \subset S_{d-2},
\]

where \(S_{d-2} \in \Gamma_{d-2}\) is the \((d - 2)\)-dimensional subspace containing one of the \((d - 2)\)-dimensional faces of \(\sigma_d\).

If \(B_{d-1}\) is elementary, it clearly satisfies the requirement of the Theorem. Hence, in the sequel we may assume that \(B_{d-1}\) is not elementary.

We consider a point \(P_0 \in \Gamma_0\) which does not lie in \(B_{d-1}\). We choose a
line $L_1$ which joins $P_0$ with an interior point of $\delta_d$ and which has no point different from $P_0$ in common with any of the $(d - 2)$-dimensional subspaces in which two hyperplanes belonging to $\Gamma_{d-1}$ intersect. (In particular, $L_1$ does then not meet any subspace belonging to $\Gamma_{d-2}$ at a point different from $P_0$.)

By assumption $\Gamma_{d-1}$ contains non-elementary hyperplanes, for instance $B_{d-1}$. Since these hyperplanes do not intersect the interior of $\delta_d$, there is at least one among them, $Q_{d-1}$ say, such that the point $Q_0$ at which it intersects $L_1$ satisfies the following condition: One of the open segments of $L_1$ determined by $P_0$ and $Q_0$ intersects neither the interior of $\delta_d$ nor any of the non-elementary hyperplanes. In other words, traversing $L_1$ from $P_0$ in that sense, or one of the senses, in which one meets non-elementary hyperplanes before meeting $\delta_d$, $Q_0$ is the first point of intersection with a non-elementary hyperplane.

The point $Q_0$ belongs to the interior of exactly one of the $(d - 1)$-dimensional cells of $\Gamma$ into which $Q_{d-1}$ is divided. The polyhedral cone $\gamma_d$ consisting of the lines joining $P_0$ with the points of this cell $\delta_{d-1}$ contains $\delta_d$ because the hyperplanes which contribute to the boundary of $\gamma_d$ belong to $\Gamma_{d-1}$ and the line $L_1 \subset \gamma_d$ intersects $\delta_d$. By the induction hypothesis, there exists in $Q_{d-1}$ an ordinary $(d - 2)$-dimensional subspace $C_0S_{d-3} \in \Gamma_{d-2}$, where

$$C_0 \in \Gamma_0, \quad C_0S_{d-3} \cap \Gamma_0 \setminus \{C_0\} \subset S_{d-3} \in \Gamma_{d-3},$$

such that

$$C_0S_{d-3} \cap \delta_{d-1} \subset S_{d-3}.$$

Putting

$$S_{d-2} = P_0S_{d-3} \in \Gamma_{d-2},$$

we consider the hyperplane

$$C_0S_{d-2} \in \Gamma_{d-1}.$$

Obviously,

(1) \quad $$C_0S_{d-2} \cap \gamma_d \subset S_{d-2},$$

hence

(2) \quad $$C_0S_{d-2} \cap \delta_d \subset S_{d-2}$$

since $\delta_d \subset \gamma_d$.

We distinguish now between two cases:

1°. In $C_0S_{d-2}$ there is outside $S_{d-2}$ no other point of $\Gamma_0$ than $C_0$. Then $C_0S_{d-2}$ is ordinary, and because of (2) this hyperplane satisfies the requirements of the Theorem.

2°. In $C_0S_{d-2}$ there is a point $A_0 \in \Gamma_0$ which does not lie in $S_{d-2}$. From the facts that
\[ C_0S_{d-2} \cap Q_{d-1} = C_0S_{d-3}, \]

\( C_0S_{d-3} \) is ordinary, and \( S_{d-3} \subset S_{d-2} \) we can conclude that \( A_0 \notin Q_{d-1} \). Further, from (1) and \( A_0 \notin S_{d-2} \) it follows that \( A_0 \notin \gamma_d \). Consequently, \( P_0A_0 \) does not meet \( \delta_{d-1} \). By Lemma 2, there exists therefore a hyperplane \( A_0T_{d-2} \in \Gamma_{d-1} \) (see fig.) such that \( T_{d-2} \) contains a \((d-2)\)-dimensional face of \( \delta_{d-1} \), and \( A_0T_{d-2} \) intersects that open segment \( P_0Q_0 \) which does not meet \( \delta_d \). From the way in which \( Q_0 \) was determined it follows that \( A_0T_{d-2} \) is elementary, thus ordinary.

It remains to be shown that

\[ A_0T_{d-2} \cap \delta_d \subset T_{d-2}. \]

The hyperplanes \( Q_{d-1} \) and \( P_0T_{d-2} \) intersect in \( T_{d-2} \). They belong to \( \Gamma_{d-1} \) and, thus, do not meet the interior of \( \delta_d \). Consequently, the closure of one of the two wedges into which they divide the space contains \( \delta_d \). Since the hyperplane \( A_0T_{d-2} \) intersects that open segment \( P_0Q_0 \) which
does not meet $\delta_d$, it can only have $T_{d-2}$ in common with the wedge containing $\delta_d$, and hence (3) holds.

This completes the proof of the Theorem.

REFERENCES