ON SETS OF VECTORS

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Given \( n \) non-void sets

\[ A_1, \ldots, A_n \]

of vectors in a finite or infinite dimensional vector space \( V \) over an arbitrary field. The letters \( a_i, b_i, c_i \) denote elements of \( A_i \), and \( \dim A_i \) denotes the dimension of the subspace spanned by \( A_i, \ i = 1, \ldots, n \). Let

\[ 0 \leq m \leq n. \]

We consider the following two statements:

- \( P_{n,m} \): \( n \) vectors \( a_1, \ldots, a_n \) always span a subspace of dimension \( \leq m \).
- \( Q_{n,m} \): There exists an integer \( h, 0 \leq h \leq m \), and an \( h \)-space containing \( h + (n - m) \) of the sets \( (1) \).

Obviously, \( Q_{n,m} \) implies \( P_{n,m} \).

The purpose of this note is to show that conversely \( P_{n,m} \) implies \( Q_{n,m} \).

This assertion is trivial for \( n = 1 \). We assume it has been proved up to \( n - 1 \). From now on let \( n > 1 \) be fixed. The case \( m = 0 \) being trivial, we may assume \( m > 0 \).

If some \( n - 1 \) of the sets \( (1) \) satisfy \( P_{n-1,m-1} \), then they satisfy \( Q_{n-1,m-1} \) by induction assumption and the sets \( (1) \) themselves will satisfy \( Q_{n,m} \). Thus we may assume that no \( n - 1 \) of the sets \( (1) \) satisfy \( P_{n-1,m-1} \), and hence, in particular, that \( m < n \).

By the last assumption, there are \( n - 1 \) vectors \( b_1, \ldots, b_{n-1} \) spanning a subspace \( V_m \) of dimension \( \geq m \). By \( P_{n,m} \), we have \( \dim V_m = m \) and every vector \( a_n \) lies in \( V_m \). This yields \( \dim A_m \leq m \); more generally,

\[ \dim A_i \leq m, \quad i = 1, \ldots, n. \]

In particular, we may assume \( V \) to be finite dimensional.

Suppose \( A_n = \{0\} \). Then \( A_1, \ldots, A_{n-1} \) satisfy \( P_{n-1,m} \). If \( m = n - 1 \), then \( Q_{n,m} \) is trivial with \( h = 0 \). If \( m < n - 1 \), then our induction assumption implies \( Q_{n-1,m} \) for the sets \( A_1, \ldots, A_{n-1} \) and the sets \( (1) \) satisfy \( Q_{n,m} \). Thus we may assume

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(2) \[ A_i \neq \{0\}, \quad i = 1, \ldots, n. \]

**Lemma.** Suppose there is an integer \( k \), \( 1 \leq k \leq m \), and there is a \( k \)-space containing \( k \) of the sets (1). Then \( Q_{n,m} \) holds.

**Proof.** Without loss of generality, we may assume that the sets

(3) \[ A_1, \ldots, A_k \]

lie in a \( k \)-space \( V_k \). We may also assume that \( k \) is minimal. Thus either \( k = 1 \) or every \( h \)-space contains fewer than \( h \) of the sets (3) if \( 0 < h < k \). Thus \( Q_{k,k-1} \) and hence \( P_{k,k-1} \) are false for the sets (3) if \( k > 1 \). There are, therefore, \( k \) linearly independent vectors

\[ c_1, \ldots, c_k. \]

They form a base of \( V_k \). (If \( k = 1 \), this remark follows directly from (2).)

We now project \( V \) parallel to \( V_k \) onto a subspace complementary to \( V_k \). Dashes denote projections. For every choice of \( a_{k+1}, \ldots, a_n \), the vectors

\[ c_1, \ldots, c_k, a_{k+1}, \ldots, a_n \]

span a space of dimension \( \leq m \). Hence the projections

\[ a'_{k+1}, \ldots, a'_n \]

always span a subspace of dimension \( \leq m - k \) and the projections

(4) \[ A'_{k+1}, \ldots, A'_n \]

satisfy \( P_{n-k,m-k} \) and hence \( Q_{n-k,m-k} \). Thus there is an integer \( g \), \( 0 \leq g \leq m - k \), and there are

\[ g + (n - k) - (m - k) = g + (n - m) \]

distinct sets

\[ A'_{i_1}, \ldots, A'_{i_{g+n-m}} \]

among the sets (4) which lie in a \( g \)-space. The \( k + g + n - m \) sets

\[ A_1, \ldots, A_k, A_{i_1}, \ldots, A_{i_{g+n-m}} \]

then lie in the \( (g+k) \)-space through \( V_k \) and that \( g \)-space. Thus our lemma is proved with \( h = g + k \).

We now complete the induction proof of \( Q_{n,m} \). The integer \( n \) was fixed. Put

\[ f = \sum_{1}^{n} \dim A_i. \]

For \( f < n \), our assertion is trivial; cf. (2). Suppose it has been proved up to \( f - 1 \). On account of the Lemma, we may assume that
\[ \dim A_n > 1. \]

Let the set \( B_n \) consist of one single element \( b_n \in A_n, b_n \neq 0 \). Thus
\[ \dim B_n = 1 < \dim A_n. \]

The sets
\[ A_1, \ldots, A_{n-1}, B_n \]
also satisfy \( P_{n,m} \). Also
\[ \sum_{1}^{n-1} \dim A_i + \dim B_n < f. \]

Hence by our induction assumption for \( f \) and by (2), there is an integer \( k, 1 \leq k \leq m \), and a \( k \)-space containing \( k + (n - m) \) of the sets (5). Thus it contains \( k + (n - m - 1) \geq k \) of the sets (1). Our lemma now yields \( Q_{n,m} \).