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MERGELYAN'S THEOREM ON UNIFORM POLYNOMIAL APPROXIMATION

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1.

Consider the set $E=[-1,1]$ on the real axis and the class $C(E)$ of continuous functions on $-1 \leqq t \leqq +1$. A proof by means of linear functionals of Weierstrass' approximation theorem would run as follows. Let $L(f)$ be a linear functional on C such that $L(x^n)=0$, $n \geqq 0$, that is

$$\int_{-1}^{+1} t^n \, d\mu(t) = 0 . \tag{1.1}$$

Multiplying by z^{-n}, $|z|>1$, we find after summation

$$F(z) = \int_E \frac{d\mu(t)}{t-z} = 0 \tag{1.2}$$

first for $|z|>1$ and then by analytic continuation for $z \notin [-1,1]$. Then by a simple residue calculation (cf. Lemma 5 below) we find $\mu \equiv 0$.

The corresponding problem for the set $E=\{z \mid |z|=1\}$ has a different structure. Here we wish to prove that a continuous function f on E with a continuous analytic extension to $|z|<1$ can be uniformly approximated by polynomials. The starting point is the same:

$$\int_{|t|=1} t^n \, d\mu(t) = 0 \tag{1.3}$$

which again yields (1.2) for $|z|>1$. But now we wish to conclude $\int f d\mu = 0$. We then need some form of the F. and M. Riesz' theorem to the effect that $d\mu = k\,d\theta$ where k is L^1-limit of polynomials. There are now rather simple proofs of this theorem (see e.g. Gårding and Hörmander [3]) but an explicit construction of an approximating polynomial is clearly simpler than any functional proof.

For a general compact set E the problem of uniform polynomial approximation was solved by Mergelyan [4].

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Mergelyan's theorem. *Let E be compact set in the plane whose complement Ω is connected. Then the set of uniform limits of polynomials on E is precisely the set of continuous functions on E which are analytic at interior points of E.*

Mergelyan's proof consists of an ingenious and delicate explicite construction which should be compared with the proof for the circle. Through the work of in particular Bishop [1], there is now also a functional proof of the theorem. Bishop's main idea was to base the proof on a corresponding result of Walsh [5] for harmonic functions. The proof in [5] is however not complete. A simplification of Bishop's proof can be obtained using ideas from Dirichlet algebras which in the general case was observed by Glicksberg and Wermer [2] and others.

In this situation it has seemed desirable to have available a self-contained direct proof of Mergelyan's theorem, based on functionals and as little function theory as possible. We shall even give a proof of this type for the existence of a solution of the Dirichlet problem. It should be stressed that the present paper contains very few new ideas; in the paper there have only been collected and adopted proofs already available in the literature. A great number of people have given contributions; the references given are by no means complete. For further references see Wermer [6].

We keep in the following the assumptions and notations from the statement of Mergelyan's theorem. The results are not always given in the most complete form, since our only goal is to obtain a proof that is as simple and clear as possible.

2.

Lemma 1. *Let α be a real measure on E. Then*

$$u(z) \equiv \int_E \log \left| \frac{1}{z-\zeta} \right| d\alpha(\zeta)$$

converges absolutely a.e. *in the plane. If $u(z)=0$ in Ω, then $u(z)=0$ in $\bar{\Omega}$ at every point of absolute convergence.*

Proof. Let $u'(z)$ be the corresponding potential generated by $|d\alpha|$. Since clearly $\int\int_{|z|<R} u'(z)\,dx\,dy<\infty$, $u'(z)<\infty$ a.e. Assume $u(z)=0$, $z \in \Omega$, and consider a point $z_0 \in \partial\Omega$ where $u'(z_0)<\infty$. To simplify the notations we take $z_0=0$. Take $\delta>0$ but small. Since $0 \in \partial\Omega$ there is a non-negative measure σ *with support in Ω* such that

$$\sigma(\{z \mid r_1 < |z| < r_2\}) = r_2 - r_1 \quad \text{for} \quad 0 < r_1 < r_2 \leqq \delta ,$$

and

$$\sigma \equiv 0 \text{ outside } |z| \leqq \delta .$$

By our assumption $u(z) = 0$, $z \in \Omega$,

$$0 = \frac{1}{\delta} \int u(z)\, d\sigma(z) = \int\limits_{|\zeta|<\varrho} + \int\limits_{|\zeta|\geqq\varrho} \left\{ \frac{1}{\delta} \int \log \left| \frac{1}{z-\zeta} \right| d\sigma(z) \right\} d\alpha(\zeta) .$$

Here $\varrho > 0$ will be kept fixed $(< \frac{1}{2})$ as $\delta \to 0$. Now clearly

$$\frac{1}{\delta} \int \log \left| \frac{1}{z-\zeta} \right| d\sigma(z) \to \log \frac{1}{|\zeta|}, \quad \delta \to 0 ,$$

uniformly in $|\zeta| \geqq \varrho$ while in $|\zeta| \leqq \varrho$

$$\begin{aligned} \frac{1}{\delta} \int \log \left| \frac{1}{z-\zeta} \right| d\sigma(z) &\leqq \frac{1}{\delta} \int \log \left| \frac{1}{|z| - |\zeta|} \right| d\sigma(z) \\ &= \frac{1}{\delta} \int_0^\delta \log \frac{1}{|r - |\zeta||} dr \\ &= \log \frac{1}{|\zeta|} + \frac{1}{\delta} \int_0^\delta \log \frac{1}{|1 - r/|\zeta||} dr \leqq \log \frac{1}{|\zeta|} + C , \end{aligned}$$

where

$$C = \sup_{T>0} \frac{1}{T} \int_0^T \log \frac{1}{|1-t|} dt .$$

Hence as $\delta \to 0$

$$\left| \int\limits_{|\zeta|\geqq\varrho} \log \frac{1}{|\zeta|} d\alpha(\zeta) \right| \leqq \int\limits_{|\zeta|\leqq\varrho} \left(\log \frac{1}{|\zeta|} + C \right) |d\alpha(\zeta)| ,$$

where by assumption the right hand side tends to zero as $\varrho \to 0$.

Lemma 2. *Let α and u be as in Lemma* 1. *If $u(z) = 0$* a.e. *in the plane, then $\alpha \equiv 0$.*

Proof. Let $g \in C^2$ with compact support. From the well-known formula

$$-\frac{1}{2\pi} \iint \Delta g(z) \log \frac{1}{|z-\zeta|} dx dy = g(\zeta)$$

it follows by integration with respect to $d\alpha(\zeta)$ that

$$0 = \int g(\zeta)\, d\alpha(\zeta)\,,$$

which yields $\alpha \equiv 0$.

LEMMA 3. (*Solution of simultaneous Dirichlet problems.*) *Let* $\varphi(z) \in C(\partial\Omega)$. *Then there is a sequence of polynomials* $P_n(z)$ *such that* $\operatorname{Re}\{P_n(z)\}$ *converges, uniformly on* E, *to* $\varphi(z)$ *on* $C(\partial\Omega)$ *and to a harmonic function at interior points of* E.

PROOF. Let α be a real measure on $\partial\Omega$ such that $\int \zeta^n d\alpha = 0$. Multiplying by z^{-n}/n, $n \geqq 1$, and summing we get a logarithmic potential $u(z)$ which vanishes in Ω. By Lemma 1 $u(z) = 0$ a.e. on $\partial\Omega$. If E has no interior points, it follows from Lemma 2 that $\alpha \equiv 0$, which proves Lemma 3 in this case. As will be still more pronounced later in the proof, the possibility of interior points causes considerable difficulties. The harmonic measure λ_a will be of fundamental importance.

Let Φ be the class of continuous functions φ on $\partial\Omega$ that admit a continuous harmonic extension U_φ to the interior points of E. By the maximum principle, Φ is closed under uniform convergence and U_φ is uniquely determined by φ. Let a be an interior point. $U_\varphi(a)$ is a positive linear functional of φ of norm 1. By the Hahn–Banach theorem, there is a measure λ_a, $\int |d\lambda_a| = 1$, so that

$$U_\varphi(a) = \int_{\partial\Omega} \varphi(\zeta)\, d\lambda_a(\zeta)\,. \tag{2.1}$$

Since $U \equiv 1$ if $\varphi \equiv 1$, $\lambda_a \geqq 0$. Apply (2.1) to $\varphi(\zeta) = \log|1/(\zeta - z)|$, where $z \in \Omega$. If δ_a is the Dirac-measure at a, (2.1) can be written $u_a(z) \equiv 0$, $z \in \Omega$, where u_a is the potential generated by $\delta_a - \lambda_a$. Furthermore, if $z_0 \in \partial\Omega$, by semi-continuity,

$$\int \log\left|\frac{1}{z_0 - \zeta}\right| d\lambda_a(\zeta) \leqq \lim_{\substack{z \to z_0 \\ z \in \Omega}} \int \log \frac{1}{|z - \zeta|}\, d\lambda_a(\zeta) = \log \frac{1}{|z_0 - a|}\,.$$

Hence $u_a(z_0)$ converges absolutely and by Lemma 1, $u_a(z) = 0$, $z \in \bar{\Omega}$.

We now return to the proof of the lemma. Since

$$\iint_{\partial\Omega} \log\left|\frac{1}{z - \zeta}\right| d\lambda_a |d\alpha| = \int |d\alpha| \log\left|\frac{1}{z - a}\right| < \infty\,,$$

λ_a vanishes on the subset of $\partial\Omega$ where the potential generated by $|d\alpha|$ diverges. Hence

$$0 = \int u(z)\,d\lambda_a(z) = \int d\alpha(\zeta) \int \log\left|\frac{1}{z-\zeta}\right| d\lambda_a(z)$$
$$= \int d\alpha(\zeta) \log\left|\frac{1}{\zeta-a}\right| = u(a)\,.$$

Hence $u(a)=0$ at all interior points of E. Since also $u(z)=0$ at a.a. boundary points, we find by Lemma 2 $\alpha\equiv 0$. Hence $\Phi = C(\partial\Omega)$, which proves Lemma 3.

3.

LEMMA 4. *Let μ be a complex measure on $\partial\Omega$. The integral*

$$F(z) = \int_{\partial\Omega} \frac{d\mu(\zeta)}{z-\zeta}$$

converges absolutely a.e. *in the plane. If $F(z)=0$ in Ω, then $F(z)=0$ in $\bar{\Omega}$ at every point of absolute convergence.* (*Compare Lemma* 1.)

PROOF. Since by Fubini's theorem

$$\iint_{|z|<R} dx\,dy \int \frac{|d\mu(\zeta)|}{|z-\zeta|} < \infty$$

the first part of the lemma is obvious.

Now assume $F(z)=0$ in Ω or, equivalently, $\int \zeta^n d\mu = 0$ and let $z_0 \in \partial\Omega$ be a point of absolute convergence of F. Following a suggestion by J. Wermer, we take a positive integer m and choose by Lemma 3 polynomials $P_m(z)$ so that

(a) $P_m(z_0)=0$,

(b) $\operatorname{Re}\{P_m(z)\} - m|z-z_0| \geqq -1, \quad z \in \partial\Omega.$

The function

$$h_m(z) = (e^{-P_m(z)} - e^{-P_m(z_0)})/(z-z_0) = (e^{-P_m(z)}-1)/(z-z_0)$$

is an entire function, and by (b)

$$|h_m(z)| \leqq \frac{1+e}{|z-z_0|}\,, \quad z \in \partial\Omega\,. \tag{3.1}$$

Finally $h_m(z) \to -(z-z_0)^{-1}$ pointwise, $z \in \partial\Omega$. We find using (3.1) and the absolute convergence of $F(z_0)$

$$0 = \int h_m(z)\,d\mu(z) \to -\int \frac{d\mu(z)}{z-z_0}, \qquad m \to \infty$$

by Lebesgue's theorem on dominated convergence. This completes the proof of Lemma 4.

LEMMA 5. *Let μ and F be as in Lemma 4. If $F(z)=0$ a.e. in the plane, then $\mu=0$. (Compare Lemma 2.)*

PROOF. Since for every $g \in C_0^2$

$$\frac{1}{\pi}\iint \frac{\partial g}{\partial \bar z}(z)\frac{1}{z-\zeta}\,dx\,dy = -g(\zeta),$$

we get by integration with respect to $d\mu(\zeta)$

$$0 = \int g(\zeta)\,d\mu(\zeta)$$

and hence $\mu \equiv 0$.

4. Proof of Mergelyan's theorem in the case of no interior points.

We assume $\int_E \zeta^n d\mu(\zeta)=0$ and shall prove $\mu\equiv 0$. The corresponding function $F(z)$ vanishes in Ω and by Lemma 4 a.e. on $\partial\Omega = E$. Hence by Lemma 5 $\mu\equiv 0$ as asserted.

We observe that this proof in fact only requires Lemma 3 with the first five lines of its proof. This case corresponds to Weierstrass' approximation theorem on $[-1,1]$.

5.

In the general case we cannot assert $\mu\equiv 0$ but need a form of the F. and M. Riesz theorem (Glicksberg and Wermer [2]). This is given in Lemmas 7 and 8.—We need a preliminary lemma concerning harmonic measures.

LEMMA 6. *Let a and b be points in the same open component c of E. Then λ_a and λ_b are absolutely continuous with respect to each other. Every measure μ on ∂E therefore has a decomposition*

$$d\mu = dh_c + d\sigma_c \tag{5.1}$$

into absolutely continuous resp. singular parts with respect to any harmonic measure in c.

PROOF. Let $u(z)\geqq 0$ be harmonic in c. Harnack's principle shows that there exists a number $K>0$, depending on a and b but independent of u, such that $K^{-1}u(a)\leqq u(b)\leqq Ku(a)$. Applying this to

$$u(z) = \int \varphi(\zeta)\, d\lambda_z(\zeta) ,$$

where $0 \leqq \varphi \in C(\partial\Omega)$, the assertion follows.

LEMMA 7. *Suppose* $\int_\Omega \zeta^n d\mu(\zeta) = 0$ *and decompose* $d\mu = dh_c + d\sigma_c$ *into absolutely continuous and singular parts with respect to some harmonic measure* λ_a. *Then separately*

$$\int \zeta^n \, dh_c = \int \zeta^n \, d\sigma_c = 0, \qquad n = 0, 1, \dots ,$$

that is,

$$\int \frac{dh_c}{z-\zeta} = 0, \qquad z \notin \bar{c} .$$

Further for $a \in c$

$$\int \frac{d\mu(\zeta)}{a-\zeta} = \int \frac{dh_c(\zeta)}{a-\zeta} \quad \text{and} \quad \int \frac{d\sigma_c(\zeta)}{a-\zeta} = 0 . \tag{5.2}$$

PROOF. We choose $a \in c$ and observe the Parseval relation

$$\int \operatorname{Re}(P)^2 \, d\lambda_a = \int \operatorname{Im}(P)^2 \, d\lambda_a \tag{5.3}$$

which holds for all polynomials with $P(a) = 0$. It follows from (2.1) applied to $\varphi(\zeta) = \operatorname{Re}\{P(\zeta)^2\}$. Let S_n be closed subsets of the support S of σ_c such that $\int_{S-S_n} |d\sigma_c| \to 0$, $n \to \infty$, and choose by Lemma 3 polynomials P_n such that

$$\operatorname{Re}(P_n(\zeta)) \geqq 2^n, \quad \zeta \in S_n, \quad \operatorname{Re}(P_n) \geqq 0, \quad \zeta \in \partial\Omega ,$$

$$\int \operatorname{Re}(P_n)^2 \, d\lambda_a \leqq 2^{-2n} .$$

By Schwarz's inequality $|\operatorname{Re}(P_n(a))| = |\int \operatorname{Re}(P_n) d\lambda_a| \leqq 2^{-n}$. We replace P_n by $P_n(z) - P_n(a)$. This implies by (5.3)

$$\sum_n \int |P_n|^2 \, d\lambda_a < \infty$$

and so

$$P_n(\zeta) \to 0 \quad \text{a.e. } (\lambda_a) .$$

Hence by bounded convergence and since $e^{-P_n} \to 0$ a.e. (σ_c),

$$0 = \int \zeta^\nu e^{-P_n(\zeta)} \, d\mu(\zeta) \to \int \zeta^\nu \, dh_c .$$

This proves the first assertion for dh_c and so for $d\sigma_c$.

For the last assertion, we observe that

$$e^{-P_n(a)}\int\frac{d\mu(\zeta)}{a-\zeta} = \int\frac{e^{-P_n(a)}-e^{-P_n(\zeta)}}{a-\zeta}\,d\mu(\zeta) + \int\frac{e^{-P_n(\zeta)}}{a-\zeta}\,d\mu(\zeta)\,.$$

Since $P_n(a)=0$ we find, letting $n\to\infty$,

$$\int\frac{d\mu(\zeta)}{a-\zeta} = \int\frac{dh_c(\zeta)}{a-\zeta}$$

which is (5.2).

LEMMA 8. *If $f(z)$ satisfies the conditions in Mergelyan's theorem and if $\int_{\partial\Omega}\zeta^n dh_c=0$ then $\int f(\zeta)\,dh_c=0$.*

PROOF. Choose $A=2\,\mathrm{Max}_E|f(\zeta)|$ and consider the branch $g(\zeta)$ of $\log(f+A)$, real for real arguments. Choose polynomials $P_n(\zeta)$ such that

$$|\mathrm{Re}\,(g-P_n)| \leqq 2^{-n}, \qquad \zeta\in\partial\Omega\,.$$

As above, we may assume $P_n(a)=g(a)$, $a\in c$, and we deduce as there

$$\lim \mathrm{Im}\,(P_n) = \mathrm{Im}\,(g) \quad \text{a.e. } (h_c)\,.$$

Hence by bounded convergence

$$\int f\,dh_c = \int (f+A)\,dh_c = \lim_{n=\infty}\int e^{P_n(\zeta)}\,dh_c = 0\,.$$

6. Proof of Mergelyan's theorem.

We consider the given function $f(\zeta)$ and polynomials as functions in $C(\partial\Omega)$. Let a functional vanish for all polynomials, that is

$$\int_{\partial\Omega}\zeta^n d\mu(\zeta)=0\,.$$

We wish to prove $\int f(\zeta)d\mu(\zeta)=0$.

Let $c_1,c_2,\ldots,c_\nu,\ldots$ be the (open) components of E. We write $d\mu=dh_{c_1}+d\sigma_{c_1}$. By Lemma 7 and 8

$$\int f dh_{c_1} = 0\,.$$

Since $\int dh_{c_1}(\zeta)/(\zeta-z)=0$ outside $\bar{c}_1$, it follows by (5.2) that

$$\int\frac{d\sigma_{c_1}}{z-\zeta} = 0, \qquad z\in c_1,\ z\in\Omega\,.$$

We make a similar decomposition $d\sigma_{c_1}=dh_{c_2}+d\sigma_{c_2}$ and find $\int f dh_{c_2}=0$ and (since $\int dh_{c_2}(\zeta)/(z-\zeta)=0$ in c_1)

$$\int \frac{d\sigma_{c_2}}{z-\zeta} = 0, \qquad z \in c_1 \cup c_2,\ z \in \Omega .$$

We obtain

$$d\mu = \sum dh_{c_\nu} + d\sigma ,$$

where the series converges in the sense of total variation and

$$\int f\, dh_{c_\nu} = 0, \qquad \nu = 1,2,\dots ,$$

and

$$\int \frac{d\sigma(\zeta)}{z-\zeta} = 0, \qquad z \in \cup\, c_\nu,\ z \in \Omega .$$

From the last relation and Lemmas 4 and 5 it follows that $\sigma \equiv 0$. Hence $\int f d\mu = 0$ which completes the proof.

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