ASYMPTOTIC ESTIMATES FOR THE
FINITE PREDICTOR

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1.

Let \( f \geq 0 \) be in \( L^1 \) of the circle group, \( \hat{f} \) its Fourier transform and \( D_n(f) \) the determinant of the \((n+1)\)-section of the Toeplitz matrix of \( f \); that is

\[
D_n(f) = \det \{ \hat{f}(i-j) \}_{i,j=0}^n.
\]

If \( \mu_n = D_n / D_{n-1} \), then it is a well known theorem of G. Szegö [4, p. 44] that

\[
\mu_n \to \mu = \exp \frac{1}{2\pi} \int_0^{2\pi} \log f(\theta) \, d\theta,
\]

where the right hand side is to be interpreted as zero if \( \log f \) is not summable. It is important to be able to estimate the rate of convergence of \( \mu_n \) in terms of smoothness properties of \( f \). Various results along this line may be found in work by G. Baxter [2], U. Grenander and M. Rosenblatt [3], U. Grenander and G. Szegö [4, § 10.10], and I. I. Hirschman, Jr. [5]. It is the purpose of this paper to give some results of a general nature which when specialized will yield the results of the above mentioned authors.

2.

Let us begin by recalling some well known facts. A more complete discussion with proofs may be found in [4]. Throughout our discussion we shall always suppose that \( f \) is a non-negative summable function with \( \log f \) also summable.

We may write \( f = |g|^2 \) where \( g \) is an outer factor in \( H^2 \). This means in particular that we can take \( \hat{g}(0) > 0 \) and if \( 1/f \in L^1 \) then \( 1/g \in H^2 \). Also, it turns out that \( \mu = \hat{g}(0)^2 \).

The quantity \( \mu_n \) is given by

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\[ \mu_n = \min \frac{1}{2\pi} \int_0^{2\pi} |p|^2 f d\theta, \]

where the minimum is taken over all \( n \)th degree polynomials,

\[ p(\theta) = \sum_0^n \hat{p}(k) e^{ik\theta}, \quad \text{with} \quad \hat{p}(0) = 1. \]

If \( u_n \) is the minimizing polynomial, then \( u_n \) may be characterized as that unique \( n \)th degree polynomial with \( \hat{u}_n(0) = 1 \) for which

\[ \int_0^{2\pi} u_n(\theta) e^{-ik\theta} f(\theta) d\theta = 0, \quad 1 \leq k \leq n. \]

There is another way to characterize \( u_n \) which will be important for what follows. If we set \( v_n = u_n/\mu_n \) then we claim that

\[ \int_0^{2\pi} |1 - \hat{g}(0)v_n g|^2 d\theta = \min \int_0^{2\pi} |1 - pg|^2 d\theta, \]

where the minimum is taken over all \( n \)th degree polynomials

\[ p(\theta) = \sum_0^n \hat{p}(k) e^{ik\theta}. \]

Indeed, the unique minimizing polynomial \( h \) is characterized by the fact that

\[ \frac{1}{2\pi} \int_0^{2\pi} \{1 - hg\} e^{-ik\theta} g d\theta = 0, \quad 0 \leq k \leq n. \]

It is not hard to check that \( \hat{g}(0)v_n \) is the polynomial with this property.

Finally we note that

\[ 1 - \mu/\mu_n = \frac{1}{2\pi} \int_0^{2\pi} |1 - \hat{g}(0)v_n g|^2 d\theta, \]

which can be checked by a direct computation.

3.

Our object in this section is to prove the following:

**Theorem 1.** (a) If \( f \in L^1 \) and \( h \) is any positive trigonometric polynomial of degree \( n \) with \( h \geq \gamma > 0 \), then
\( 1/\mu - 1/\mu_n \leq \frac{\nu}{2\pi \gamma} \int_0^{2\pi} |1/f - h|^2 f \, d\theta, \)

where

\[ \nu = \exp \frac{1}{2\pi} \int_0^{2\pi} \log h \, d\theta. \]

(b) If \( f \geq \alpha > 0 \) then

\( \alpha \sum_{k>n} |(1/g) \hat{(k)}|^2 \leq 1 - \mu/\mu_n. \)

(c) If \( 0 < \alpha \leq f \leq \beta < \infty \) and if \( s_n = \sum_0^n (1/g) \hat{(k)} e^{i k \theta} \) with \( |s_n|^2 \leq \gamma < \infty \) for all \( n \), then

\( \frac{\alpha^2}{2(1 + \beta \gamma)} \sum_{|k|>n} |(1/f) \hat{(k)}|^2 \leq 1 - \mu/\mu_n. \)

**Proof.** To prove (a) we first write \( h = |p|^2 \), where \( p(\theta) = \sum_0^n \hat{p}(k) e^{i k \theta} \), \( \hat{p}(0) > 0 \) and \( 1/p \in H^2 \). This is just the well known Fejér–Riesz theorem on the factorization of non-negative trigonometric polynomials. Hence, recalling that \( f = |g|^2 \), we get

\[
\frac{1}{2\pi} \int_0^{2\pi} |1/f - h|^2 f \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} |p|^2 |1/\hat{p} g - pg|^2 \, d\theta
\]

\[
\geq \frac{\gamma}{2\pi} \int_0^{2\pi} |1/\hat{p} g - 1/\hat{p} g(0) + 1/\hat{p} g(0) - pg|^2 \, d\theta.
\]

Since \( 1/(pg) \in H^2 \), it follows that \( 1/\hat{p} g - 1/\hat{p} g(0) \) is orthogonal to \( 1/\hat{p} g(0) - pg \). Therefore,

\[
\frac{1}{2\pi} \int_0^{2\pi} |1/f - h|^2 f \, d\theta \geq \frac{\gamma}{2\pi} \int_0^{2\pi} |1/\hat{p} g(0) - pg|^2 \, d\theta
\]

\[
\geq \frac{\gamma}{2\pi \nu \mu} \int_0^{2\pi} |1 - \hat{g}(0) pg|^2 \, d\theta
\]

\[
\geq \frac{\gamma}{2\pi \nu \mu} \int_0^{2\pi} |1 - \hat{g}(0) v_n g|^2 \, d\theta = \left[1/\mu - 1/\mu_n\right] \gamma/\nu.
\]

This gives our inequality in (a).

To prove (b) we have simply
\[ 1 - \mu/\mu_n = \frac{1}{2\pi} \int_0^{2\pi} |1 - \hat{g}(0) v_n|^2 d\theta \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} |g|^2 |1/g - \hat{g}(0) v_n|^2 d\theta \geq \sum_{k > n} |(1/g)(k)|^2. \]

Finally, to prove (c) we have
\[ \sum_{|k| > n} |(1/f)(k)|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |1/f - |s_n|^2 f| d\theta \]
\[ \leq \frac{1}{2\pi} \int_0^{2\pi} |1/f - |s_n|^2 f| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |1/g - |s_n|^2 g| d\theta \]
\[ \leq \frac{1}{\pi} \left\{ \int_0^{2\pi} |1/g - |s_n|^2 d\theta + \int_0^{2\pi} |g s_n|^2 |1/g - |s_n|^2 d\theta \right\} \]
\[ \leq \frac{1 + \beta x}{\pi} \int_0^{2\pi} |1/g - \hat{g}(0) v_n|^2 d\theta \]
\[ \leq \frac{1 + \beta x}{\pi} \int_0^{2\pi} |1/g - \hat{g}(0) v_n|^2 f d\theta = (1 - \mu/\mu_n) 2(1 + \beta y)/\alpha^2. \]

4.

We now want to indicate how we can use the previous elementary estimates to obtain most of the results of the previously mentioned authors. We start with the sufficiency part of a result of Grenander and Rosenblatt [3] (see also [4; § 10.10]).

If \( f \) has no zeros and its periodic extension is real analytic, then
\[ \delta_n = \mu_n - \mu = O(\varepsilon^n), \]
where \( 0 \leq \varepsilon < 1. \)

Since \( f \) has no zeros, the periodic extension of \( 1/f \) is analytic. This means that \( 1/f \) may be considered to be an analytic function on the circle. Indeed, let \( \log w \) be any determination of the logarithm function and set \( F(w) = 1/f(-i \log w) \) for \( w = e^{i\theta} \). The analyticity of \( 1/f \) implies that \( F \) may be extended to be analytic in an open annulus \( \{z: \varepsilon < |z| < 1/\varepsilon\} \) with \( 0 \leq \varepsilon < 1 \). We can expand \( F \) in a Laurent expansion about zero to get
\[ F(z) = \sum_{n=-\infty}^{\infty} (1/f)(n) z^n, \]
where
\[ F_1(z) = \sum_{n=\infty}^{0} (1/f)^{(n)} z^n \text{ converges for } |z| > \varrho, \]

\[ F_2(z) = \sum_{n=0}^{\infty} (1/f)^{(n)} z^n \text{ converges for } |z| < 1/\varrho. \]

It follows from this that the symmetric partial sums of $1/f$ converge uniformly to $1/f$ and we may apply theorem 1 (a) to get

\[ \delta_n = \mu_n - \mu = O \left( \sum_{|k|>n} |(1/f)^{(k)}|^2 \right). \]

But $(1/f)^{(k)} = O(\varrho^k)$ implies $\delta_n = O(\varrho^n)$.

Our second example is a result due to Baxter [2].

If $f > 0$ and $\sum |\hat{f}(k)||k|^2 < \infty$, $\lambda \geq 0$, then $\delta_n = o(n^{-2\lambda})$.

For fixed $\lambda$ the functions in this class form a Banach algebra with spectrum the unit circle and hence $f > 0$ implies that $1/f$ is in this algebra. If $n$ is sufficiently large we get

\[ n^{2\lambda} \sum_{|k|>n} |(1/f)^{(k)}|^2 \leq \sum_{|k|>n} |k|^{2\lambda} |(1/f)^{(k)}|^2 \leq \sum_{|k|>n} |k|^3 |(1/f)^{(k)}|. \]

The result is now an immediate consequence of theorem 1 (a).

Finally we give an example of a result due to I. I. Hirschman, Jr. [5].

If $0 < \alpha \leq f \leq \beta < \infty$ and $\lambda > 1$, then $\delta_n = o(n^{-\lambda})$ if and only if

\[ n^{\lambda} \sum_{|k| \geq n} |\hat{f}(k)|^2 = O(1). \]

The functions of this class form a subalgebra of the class of summable Fourier series. Under a suitable norm they form a Banach algebra with spectrum the unit circle [5]. Hence the result now follows from Theorem 1, since $f$ is in the algebra if and only if $1/f$ is in the algebra.

**Remarks.** (a) Let $s_n = \sum_0^n (1/g)^{(k)} e^{ik\theta}$; then under the hypothesis of Theorem 1(b) we have

\[ \frac{1}{2\pi} \int_0^{2\pi} |s_n - \hat{g}(0)v_n|^2 \, d\theta = O(\delta_n). \]

Indeed, from (4) we may write

\[ \{1 - \mu/\mu_n\}^t = \left[ \frac{1}{2\pi} \int_0^{2\pi} |g|^2 |1/g - \hat{g}(0)v_n|^2 \, d\theta \right]^t \geq \alpha^t \left[ \frac{1}{2\pi} \int_0^{2\pi} |s_n - \hat{g}(0)v_n|^2 \, d\theta \right]^t - \alpha^t \left\{ \sum_{|k|>n} |(1/g)^{(k)}|^2 \right\}^t. \]
Apply Theorem 1 (b) and we have our result. This result was obtained in special cases by Baxter [2] and Hirschman [5].

(b) If \( f = |g|^2 \), \( g \) outer in \( H^2 \), it is in general an undecided question as to which smoothness properties of \( f \) carry over to \( g \). Our Theorem 1 sheds a small amount of light on this problem. For example, if \( f \in \text{Lip}(\lambda, 2) \), then it is a well known result [1; p. 171] that

\[
\sum_{|k| \leq n} |\hat{f}(k)|^2 = O(n^{-2\lambda}).
\]

The converse is also true. Hence if \( 0 < \alpha \leq f \leq \beta < \infty \) and if \( s_n \) are the partial sums of the Fourier expansion of \( f \) with \( s_n \geq \gamma > 0 \), then an application of Theorem 1 (a) tells us \( \delta_n(1/f) = O(n^{-2\lambda}) \) which in turn, by Theorem 1 (b) tells us that

\[
\sum_{k \geq n} |\hat{g}(k)|^2 = O(n^{-2\lambda}).
\]

This means we also have \( g \in \text{Lip}(\lambda, 2) \).

5.

We would now like to sharpen and complete the results we have previously obtained. We shall show that if \( \delta_n = \mu_n - \mu \) goes to zero sufficiently rapidly, then \( f^{-1} \) has a summable Fourier series. Specifically we shall prove the following:

**Theorem 2.** If \( \sum_{k=0}^{\infty} \{2^k \delta_{2^k}\}^{1/4} < \infty \) and \( \log f \) is summable, then \( 1/f \) has a summable Fourier series.

Roughly speaking this result says that, unless \( f \) has no zeros and is very smooth most of the time, then \( \delta_n \) cannot go to zero very much faster than \( 1/n \). This is to be compared with the results of the next section. Note that if \( f \geq \alpha > 0 \), then theorem 2 is an immediate consequence of theorem 1 (b). Indeed we get

\[
\sum_{2^{n+1}}^{2^{n+1}} |(1/g)^\wedge(k)| \leq 2^{1/n} \left[ \sum_{2^{n+1}}^{2^{n+1}} |(1/g)^\wedge(k)|^2 \right]^{1/2} \leq \alpha^{-1/2} \delta_{2n}^{1/2}.
\]

Summing both sides over \( n \) we get the result.

In case \( f \) is bounded above, it is not hard to show that theorem 2 is a consequence of Theorem 1.1 of Baxter [2a]. Indeed, the general case can be obtained by an application of an idea developed in this same paper [2a]. This was pointed out to us by I. I. Hirschman.

For the sake of completeness we shall briefly review this material (see also [5]). Let \( H \) be the Hilbert space generated by the one-sided trigonometric polynomials \( p(\theta) = \sum_o \hat{p}(k) e^{ik\theta} \) in the \( L^2 \) norm given by the
measure $f d \theta$, and let $H_n$ be the subspace generated by polynomials of
degree $n$. Using the notation of our previous sections, we find that the
polynomial $v_n - v_{n-1}$ is in the one-dimensional space $H_n \ominus H_{n-1}$. It is
a simple matter to check that the polynomial $e^{i\theta \beta \bar{v}_n}$ also is in this latter
space. Hence, there is a constant $\alpha_n$ so that

$$v_n - v_{n-1} = \alpha_n e^{i\theta \beta \bar{v}_n},$$

and therefore

$$v_n = \sum_{k=0}^n \alpha_k e^{i\theta \beta \bar{v}_k}.$$

Now, the polynomials $\mu_n e^{i\theta \beta \bar{v}_n}$ are the orthonormal Szegő polynomials
associated with $f$ and hence from (3) and (4) we get

$$1 - \mu/\mu_n = \mu \sum_{k=n+1}^{\infty} |\alpha_k|^2/\mu_k.$$

Proof of Theorem 2. Let $\| \cdot \|_1$ be the $l^1$ norm; i.e. for any function $h$
with summable Fourier series we write

$$\|h\|_1 = \sum_{-\infty}^{\infty} |\hat{h}(k)|.$$

From (8) we get $\|v_n\|_1 - \|v_{n-1}\|_1 \leq |\alpha_n| \|v_n\|_1 \leq \|v_n\|_1 + \|v_{n-1}\|_1$. Consequently,

$$|1 - |\alpha_n|| \|v_n\|_1 \leq \|v_{n-1}\|_1,$$

and repeated iteration of this result gives

$$\left| \prod_{m=1}^n (1 - |\alpha_k|) \right| \|v_n\|_1 \leq \|v_m\|_1.$$

From the fact that $\sum (2^k \delta_{y_k})i \leq \infty$, it follows from (9), using exactly the
same kind of argument as used after the statement of theorem 2, that
$\sum |\alpha_k| < \infty$. Therefore $\sum (1 - |\alpha_k|)$ converges and the sequence $\{|v_n|_1\}$
is uniformly bounded by a constant $C$.

Returning to (8) we see that

$$\|v_{n+p} - v_n\|_1 \leq C \sum_{n+1}^{n+p} |\alpha_k|$$

and hence $\{v_n\}$ is Cauchy in the $l^1$ norm. From (4) it follows immediately
that the limit of this sequence in the $l^1$ norm is $[g(0)g]^{-1}$. Therefore,
$1/g$ and hence $1/f = 1/|g|^2$ have summable Fourier series.

Remarks (a) The result we have just obtained contains the necessity
part of the Grenander--Rosenblatt result. Indeed, if $\delta_n = O(g^n)$, $0 \leq g < 1$,
then \( f \) is bounded away from zero and we may apply theorem 1 (b) to show that \( (1/g)^{-}(n) = O(g^n) \). This, in turn, shows that the periodic extension of \( 1/g \) is an analytic function of \( \theta \) and hence the periodic extension of \( 1/f = 1/|g|^2 \) is an analytic function of \( \theta \). Now, \( 1/f \) can have no zeros since this would preclude the possibility of \( f \) being summable. Hence \( f \) is real analytic with no zeros.

(b) The statement \( \sum_{0}^{\infty} \{2^k \delta_{2^k}\}^{\frac{1}{2}} < \infty \) is clearly equivalent with the statement \( \sum_{0}^{\infty} \{\delta_{n}/n\}^{\frac{1}{2}} < \infty \). Following the lead of Hirschman [5], it is natural to conjecture that the class of functions which satisfy the condition

\[
\|h\|_2 = \sum_{n=0}^{\infty} \left\{ \sum_{|k| \geq 2^n} |\hat{h}(k)|^2 \right\}^{\frac{1}{2}} < \infty
\]

is a Banach algebra under the norm

\[
\|h\| = c\{\|h\|_1 + \|h\|_2\},
\]

where \( c \) is a suitably chosen constant and \( \|\cdot\|_1 \) is the \( l^1 \) norm. This is indeed the case and moreover, the spectrum of this algebra is the unit circle. The proof requires only a slight modification of the proof given by Hirschman in a special case. It is easy to see that one gets an equivalent norm by taking

\[
\|h\|_2 = \sum_{n=1}^{\infty} \left\{ \sum_{|k| \geq n} |\hat{h}(k)|^2 \right\}^{\frac{1}{2}}.
\]

One interest in knowing that we have a Banach algebra stems from the possibility of being able to get asymptotic estimates for \( \delta_n \) in terms of \( f \) rather than in terms of \( 1/f \).

6.

It was pointed out by Grenander and Rosenblatt [3] that if \( f \) has zeros, then in general we cannot expect \( \delta_n \) to go to zero faster than \( 1/n \). As they pointed out, a function for which \( \delta_n \) goes to zero at precisely this rate is \( f(\theta) = |1 - e^{i\theta}|^2 \). It is the purpose of this section to generalize their results. If for any non-negative \( f \) with \( \log f \) summable we set \( \delta_n(f) = 1 - \mu/\mu_n \) then we have the following:

**Theorem 3.** If \( f = |e^{i\theta} - 1|^2 \), where \( f \) and \( f_1 \) are non-negative and summable and \( \log f_1 \) is summable, then

\[
\delta_n^{\frac{r}{s}}(f) \leq (1/r)^{\frac{r}{s}} + 2\delta_s^{\frac{r}{s}}(f_1), \quad r + s = n.
\]

**Proof.** Let \( f = |g|^2 \) and \( f_1 = |g_1|^2 \) where \( g \) and \( g_1 \) are outer factors in \( H^2 \). From (4) we have
\[ \partial_n(f) = \frac{1}{2\pi} \int_0^{2\pi} |1 - \hat{g}(0)v_ng|^2 \, d\theta . \]

Let \( w_s \) be the polynomial such that

\[ \partial_s(f_1) = \frac{1}{2\pi} \int_0^{2\pi} |1 - \hat{g}_1(0)w_sg_1|^2 \, d\theta , \]

and set

\[ p_r(\theta) = \sum_{k=0}^{r-1} (1 - k/r) e^{ikt\theta} . \]

From (3) it follows that \( v_n \) has a minimizing property and hence

\[
\partial_n(f) \leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 - p_r(1 - e^{i\theta})\hat{g}_1(0)w_s g_1|^2 \, d\theta \right\}^{\frac{1}{2}} \\
\leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 - p_r(1 - e^{i\theta})|^2 \, d\theta \right\}^{\frac{1}{2}} + \\
+ \frac{1}{2\pi} \left\{ \int_0^{2\pi} |p_r(1 - e^{i\theta})|^2 |1 - \hat{g}_1(0)w_sg_1|^2 \, d\theta \right\}^{\frac{1}{2}} .
\]

Now, it is easily computed that

\[ p_r(1 - e^{i\theta}) = 1 - (1/r) \sum_{k=1}^{r} e^{ikt\theta} . \]

Therefore,

\[ |p_r(1 - e^{i\theta})| \leq 2 , \]

\[ \frac{1}{2\pi} \int_0^{2\pi} |1 - p_r(1 - e^{i\theta})|^2 \, d\theta = 1/r . \]

If we use these estimates in (11) we get our result.

**Corollary.** If \( f = f_1 \prod_{j=1}^{k} |e^{i\theta} - e^{i\theta_j}|^{2\lambda_j} \), and \( \lambda = \sum \lambda_j \), then

\[ \delta_n(f) = O(1/n + \delta_{(n/2s)}(f_1)) . \]

This is obtained by iterating (10) \( \lambda \) times. At the first stage choose \( r = [(n + 1)/2] \) and \( s = [n/2] \), say, and then continue in this way with \( \delta_{(n/2s)} \).
REFERENCES


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