MATH. SCAND. 15 (1964), 93–96

STRONGLY SUBHARMONIC FUNCTIONS

LARS GÅRDING and LARS HÖRMANDER

The purpose of this note is to give a short proof of the F. and M. Riesz theorem [1] by proving a corresponding fact concerning subharmonic functions in any number of variables. Our result could also be used to prove various generalizations of the Riesz theorem, such as those given by Stein and Weiss [2], but we shall only give one such application here.

Let $\Omega=\{x;\ x\in R^n, |x|^2=x_1{}^2+\ldots+x_n{}^2<1\}$ be the unit ball in R^n, let $d\omega$ be the normalized orthogonally invariant measure on $\partial\Omega=\{x;\ |x|=1\}$ and let $P(x,\omega)=(1-|x|^2)/|x-\omega|^n$ be the Poisson kernel of Ω. Then $P(x/r,\omega)$ is the Poisson kernel of the ball $|x|<r$. Let u be a subharmonic function in Ω. We first recall some simple classical facts.

(i) The smallest harmonic majorant of u in $\{x;\ |x|<r<1\}$ is

$$h_r(x) = \int P(x/r,\omega)\, u(r\omega)\, d\omega \tag{1}$$

and increases with r. The smallest harmonic majorant in Ω is $h(x)=\lim h_r(x)$. By Harnack's theorem, h is finite if and only if the mean values

$$h_r(0) = M_r(u) = \int u(r\omega)\, d\omega$$

are bounded and then $h(0)-h_r(0)\to 0$, that is

$$\int \big(h(r\omega)-u(r\omega)\big)\, d\omega \to 0, \quad r\to 1\,. \tag{2}$$

(ii) If

$$\int u^+(r\omega)\, d\omega \leqq C < \infty, \qquad 0\leqq r<1\,,$$

where $u^+=\max(u,0)$, then u has a finite harmonic majorant by (i) and

$$\int |u(r\omega)|\, d\omega$$

is bounded since $M_r(|u|)=2M_r(u^+)-M_r(u)$ and $M_r(u)$ increases. Passing to the limit in (1) we see that there exists a measure $d\mu$ on $\partial\Omega$ such that

Received November 20, 1964.

$$h(x) = \int P(x,\omega)\, d\mu(\omega) .$$

Let $g(\omega)$ be continuous and put $f(x) = \int P(x,\omega) g(\omega) d\omega$. Since $P(r\omega,\omega') = P(r\omega',\omega)$, we have

$$\int h(r\omega)\, g(\omega)\, d\omega = \int f(r\omega)\, d\mu(\omega) ,$$

where the right side tends to $\int g(\omega) d\mu(\omega)$ as $r \to 1$. Hence $d\mu(\omega)$ is the weak limit of the measures $h(r\omega) d\omega$ and by (2) also of $u(r\omega) d\omega$. In particular, $d\mu$ is unique. We say that $d\mu$ is the boundary measure of u (and h). Notice that $d\mu$ increases with u. Applying this to the subharmonic function $|h(x)| \leqq \int P(x,\omega)\,|d\mu(\omega)|$, we see that $|d\mu|$ majorizes the boundary measure $d\nu$ of $|h|$. Since for continuous g we have

$$|\mu(g)| = \lim \left| \int h(r\omega) g(\omega)\, d\omega \right| \leqq \lim \int |h(r\omega)|\; |g(\omega)|\, d\omega = \nu(|g|),$$

we also have $|d\mu| \leqq d\nu$ so that $|d\mu| = d\nu$. Hence $|d\mu|$ is the boundary measure of $|h|$ and by (2) also of $|u|$. If we form the Lebesgue decomposition

$$d\mu = \lambda(\omega)\, d\omega + d\mu_s(\omega) , \tag{3}$$

where λ is summable and $d\mu_s$ singular with respect to $d\omega$, then

$$h(r\omega) \to \lambda(\omega), \quad r \to 1 , \tag{4}$$

for almost all ω. Further,

$$\int |u(r\omega) - \lambda(\omega)|\, d\omega \to \int |d\mu_s(\omega)|, \quad r \to 1 . \tag{5}$$

In fact, let $f(x)$ be the Poisson integral of the measure $\lambda(\omega) d\omega$. Then (5) holds with $\lambda(\omega)$ replaced by $f(r\omega)$ and an approximation of λ by continuous functions shows that

$$\int |f(r\omega) - \lambda(\omega)|\, d\omega \to 0, \quad r \to 1 .$$

Our purpose is to give conditions which guarantee that the boundary measure of a subharmonic function is absolutely continuous. Before stating our result we recall the following simple consequence of Jensen's inequality:

(iii) If u is subharmonic, if φ is convex and increasing on R and $\varphi(-\infty) = \lim \varphi(t)$ for $t \to -\infty$, then $\varphi(u)$ is subharmonic.

THEOREM. *Let u be subharmonic in Ω, let φ be a non-negative convex increasing function on R such that $\varphi(t)/t \to \infty$ as $t \to \infty$, $\varphi(t) \to \varphi(-\infty)$ as $t \to -\infty$ and assume that*

$$\int \varphi(u(r\omega))\, d\omega \leqq C < \infty, \qquad 0 \leqq r < 1 . \tag{6}$$

Then u has a boundary measure (3) *with $d\mu_s \leqq 0$ and the boundary measure of $\varphi(u)$ is absolutely continuous and equals $\varphi(\lambda(\omega))d\omega$. In particular,*

$$\lim_{r\to 1} \int |\varphi(u(r\omega)) - \varphi(\lambda(\omega))|\, d\omega = 0 . \tag{7}$$

REMARK. One might say that the function $\varphi(u)$ is more subharmonic than u. This motivates the following definition. A subharmonic function $v \geqq 0$ is said to be strongly subharmonic if $\varphi^{-1}(v(x))$ is subharmonic for some φ as described above. Examples: if $f(z)$ is analytic, then $\log|f(z)|$ is subharmonic and hence $|f(z)|$ is strongly subharmonic. If $u_1(x), \ldots, u_n(x)$ is a conjugate system of real harmonic functions, i.e. satisfy $\sum \partial u_j/\partial x_j = 0$, $\partial u_j/\partial x_k = \partial u_k/\partial x_j$ for $j,k = 1,\ldots,n$, then $|u| = (u_1^2 + \ldots + u_n^2)^{\frac{1}{2}}$ is strongly subharmonic. In fact, it is proved in [2] that $|u|^p$ is subharmonic for $p \geqq (n-2)/(n-1)$. The theorem may be expressed as follows: if $v(x)$ is strongly subharmonic and has a finite harmonic majorant in $|x| < 1$, then its boundary measure is absolutely continuous. This explains the title of the paper.

PROOF. (6) implies that $M_r(u^+)$ is bounded and hence u has a boundary measure $d\mu$. Let $E \subset \partial\Omega$ be open with measure $m(E)$ and let $0 \leqq g(\omega) \leqq 1$ be continuous and vanish outside E. Put $\alpha(s) = \sup t/\varphi(t)$ for $t \geqq s > 0$. Then

$$\int u(r\omega) g(\omega)\, d\omega \leqq \alpha(s) \int \varphi(u(r\omega)) g(\omega)\, d\omega + s \int g(\omega)\, d\omega$$

so that, by (6),

$$\int g(\omega)\, d\mu(\omega) \leqq C\alpha(s) + s\, m(E) .$$

Putting $s = (m(E))^{-\frac{1}{2}}$, the right side tends to zero with $m(E)$. Hence $d\mu_s \leqq 0$. Let $d\nu$ be the boundary measure of $\varphi(u)$. Then

$$\lim_{r\to 1} \int \varphi(u(r\omega)) g(\omega)\, d\omega = \int g(\omega)\, d\nu(\omega) \tag{8}$$

for every continuous g. By virtue of (2) and (4), there exists a sequence $r_j \to 1$ such that $u(r_j\omega) \to \lambda(\omega)$ almost everywhere. Hence, taking $g \geqq 0$ in (8) and applying Fatou's lemma, we conclude that

$$dv(\omega) \geqq \varphi(\lambda(\omega))\, d\omega \,. \tag{9}$$

In particular, $\varphi(\lambda(\omega))$ is summable. Since $d\mu_s \leqq 0$,

$$u(x) \leqq \int P(x, \omega)\, \lambda(\omega)\, d\omega \,,$$

so that, by Jensen's inequality,

$$\varphi(u(x)) \leqq \int P(x, \omega)\, \varphi(\lambda(\omega))\, d\omega \,.$$

This implies the inequality opposite to (9). Since (7) follows from (5) applied to $\varphi(u(x))$, the proof is finished.

The theorem implies the following theorem of F. and M. Riesz [1]: if $f(z)$ is analytic in the unit disc and $\int |f(re^{i\theta})|\, d\theta$ is bounded then f has an absolutely continuous boundary measure $d\mu(\theta)$. In particular, $f(e^{i\theta}) = \lim f(re^{i\theta})$, $r \to 1$, exists for almost all θ, $d\mu(\theta) = f(e^{i\theta})\, d\theta$ and

$$\int |f(re^{i\theta}) - f(e^{i\theta})|\, d\theta \to 0, \quad r \to 1 \,.$$

In fact, $|f(z)|$ is strongly subharmonic and has a finite harmonic majorant so that its boundary measure dv is absolutely continuous. Since $dv \geqq |d\mu|$ (actually $dv = |d\mu|$), $d\mu$ is also absolutely continuous.

It follows in the same way that if $u = (u_1, \ldots, u_n)$ is a conjugate system of harmonic functions in Ω and $|u|$ has a finite harmonic majorant in Ω, then the boundary measure $d\mu$ of u is absolutely continuous, $d\mu = u(\omega) d\omega$, and $\int |u(r\omega) - u(\omega)|\, d\omega \to 0$ as $r \to 1$. This is essentially the basic result of [2].

REFERENCES

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UNIVERSITY OF LUND, SWEDEN
AND
INSTITUTE FOR ADVANCED STUDY, PRINCETON, N.J., U.S.A.