DIOPHANTINE EQUATIONS IN RECURSIVE DIFFERENCE

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We determine the general solutions (in natural numbers) of a variety of linear equations involving the recursive difference function $x \div y$ which has the value 0 when $x \leq y$ and is equal to the excess of $x$ over $y$ when $x > y$. We assume a familiarity with properties of recursive difference, in particular the properties

$$x + (y \div x) = y + (x \div y), \quad x \div (x \div y) = y \div (y \div x),$$

$$(x + y) \div z = (x \div z) + \{y \div (z \div x)\},$$

$$x = (x \div y) + \{x \div (x \div y)\}$$

$$x(y \div z) = xy \div xz.$$

To illustrate the kind of results to be obtained, we consider first the equation

$$(1) \quad x + (y \div x) = a.$$  

The general solution of this equation is

$$(1*) \quad \left\{ \begin{array}{l}
x = a \div (u \div v) \\
y = a \div (v \div u)
\end{array} \right.$$  

where $u, v$ are arbitrary parameters. For if $x, y$ are given by (1*) then

$$x + (y \div x) = \{a \div (u \div v)\} + \{(a \div (v \div u)) \div (a \div (u \div v))\}$$

$$= a$$

(consider in turn the cases $u \leq v, u > v$), so that equation (1) is satisfied. Conversely, since

$$x = \{x + (y \div x)\} \div (y \div x)$$

and

$$y = \{x + (y \div x)\} \div (x \div y)$$

identically, therefore any solution $x, y$ of (1) may be obtained from (1*)

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giving \( u \) the value \( y \) and \( v \) the value \( x \). Similarly, the general solution of the equation

\[(2) \quad x \div (x \div y) = a \]

is

\[(2^*) \quad \begin{cases} x = a + (u \div v) \\ y = a + (v \div u) \end{cases} \]

We consider next the equation

\[(3) \quad x \div y = a . \]

The general solution of (3) is

\[(3^*) \quad x = a + \{y \div (1 \div a)u\} . \]

For if \( x \) satisfies \( 3^* \) then

\[
x \div y = \left[\{y \div (1 \div a)u\} \div y\right] + \left[a \div \{y \div (y \div (1 \div a)u)\}\right]
= a \div \{y \div (y \div (1 \div a)u)\}
= a
\]

(consider in turn the cases \( a = 0, a \geq 1 \)) so that all \( x \) given by \( 3^* \) satisfy (3). Conversely, since

\[
x = (x \div y) + \{y \div (1 \div (x \div y))(y \div x)\}
\]

therefore any solution \((x, y)\) of (3) may be obtained from \( 3^* \) giving \( u \) the value \( y \div x \).

It follows that the general solution of

\[(3.1) \quad x_1 \div (x_2 \div (x_3 \div \ldots \div (x_n \div u_n) \ldots)) = u_0 \]

is

\[(3.2) \quad x_r = u_{r-1} + \{u_r \div (1 \div u_{r-1})v_r\}, \quad 1 \leq r \leq n , \]

where \( u_1, u_2, \ldots, u_{n-1}, v_1, v_2, \ldots, v_n \) are arbitrary parameters. For if we write \( u_{n-1} \) for \( x_n \div u_n, u_{n-2} \) for \( x_{n-1} \div u_{n-1} \) and so on up to \( u_1 \) for \( x_2 \div u_2 \) then also \( x_1 \div u_1 = u_0 \), so that

\[
x_r \div u_r = u_{r-1}, \quad 1 \leq r \leq n ,
\]

and by \( 2^* \) the general solution of this system of equations is \( 3.2 \).

We turn next to the equation

\[(4) \quad x \div a = y \div b \]

of which the general solution is

\[(4^*) \quad y = (x \div a) + \{b \div (1 \div (x \div a))u\} . \]
We omit the verification that (4*) satisfies (4). That every solution of (4) is contained in (4*) follows from the identity
\[ y = (y - b) + \{ b - (1 - (y - b))(b - y) \} \]
which shows that if \((x, y)\) is solution of (4) then this \(y\) may be obtained from (4*) by giving \(u\) the value \(b - y\).

Similarly, the equation
\[(5)\]
\[ a - x = a - y \]
has the general solution
\[(5*)\]
\[ y = \{ a - (a - x) \} + \{ 1 - (a - x) \} u . \]
The verification that (5*) satisfies (5) is trivial. The generality of the solution follows from the identity
\[ y = \{ a - (a - y) \} + \{ 1 - (a - y) \} (y - a) . \]

The solution of the apparently more general equation
\[(6)\]
\[ a - x = b - y \]
is readily derived from (5*). For we may suppose, without loss of generality, that \(a \leq b\) and so \(a = b - (b - a)\) whence
\[ b - y = (b - (b - a)) - x = b - (x + (b - a)) \]
of which the general solution is (by (5*))
\[(6*)\]
\[ y = \left[ b - \{ b - (x + (b - a)) \} \right] + \left[ 1 - \{ b - (x + (b - a)) \} \right] u \\
= \{ b - (a - x) \} + \{ 1 - (a - x) \} u . \]

Although the equation
\[(7)\]
\[ x + y = a \]
involves only elementary addition its general solution in natural numbers is
\[(7*)\]
\[ \begin{cases} x = a - u \\ y = a - (a - u) \end{cases} \]
and so depends upon the recursive difference function. That (7*) is a solution of (7) for any value of \(u\) follows from the identity
\[(7.1)\]
\[ (a - u) + \{ a - (a - u) \} = a \]
and that (7*) is the general solution is shown by the identity
\[ \{(x + y) - y\} + [(x + y) - \{(x + y) - y\}] = x + y \]
which reveals that any pair \((x, y)\) which satisfy (7) may be obtained from \((7^*)\) by giving \(u\) the value \(y\).

From 7.1 we readily derive the general solution of

\[
x_1 + x_2 + \ldots + x_{n+1} = \alpha.
\]

Write \(s_0 = 0, s_{k+1} = s_k + u_k\) so that

\[
s_k = u_1 + u_2 + \ldots + u_k
\]

where \(u_1, u_2, \ldots, u_n\) are arbitrary parameters, then the general solution of (8) is

\[
\begin{align*}
(x_k &= (a \div s_{k-1}) \div (a \div s_k), \quad 1 \leq k \leq n, \\
x_{n+1} &= a \div s_n.
\end{align*}
\]

For by (7.1)

\[
(a \div s_k) + [(a \div s_{k-1}) \div (a \div s_k)] = a \div s_{k-1}, \quad 1 \leq k \leq n,
\]

and so, by addition,

\[
\sum_{k=1}^{n} [(a \div s_{k-1}) \div (a \div s_k)] + (a \div s_n) = a
\]

which proves that \((8^*)\) is a solution of (8) for all \(u_1, u_2, \ldots, u_n\). Conversely, writing \(X_{k+1} = X_k + x_k, X_0 = 0\), we have

\[
\{X_{n+1} - X_k\} \div \{X_{n+1} - X_{k+1}\} = x_{k+1}, \quad 0 \leq k \leq n,
\]

so that any set of values \((x_1, x_2, \ldots, x_{n+1})\) which satisfy (8) may be obtained from \((8^*)\) by giving \(u_k\) the value \(x_k, 1 \leq k \leq n\).

As a final and rather more difficult example we consider the equation

\[
(x_0 \div x_1) + (x_1 \div x_2) + \ldots + (x_n \div x_0) = \alpha.
\]

The particular case of (9), with \(n = 1\),

\[
(x \div y) + (y \div x) = \alpha
\]

has the general solution

\[
x = u + a(1 \div v), \quad y = u + a\{1 \div (1 \div v)\}
\]

since

\[
\{u + a(1 \div v)\} \div [u + a\{1 \div (1 \div v)\}] = a(1 \div v)
\]

and, if \(\min(x, y) = x \div (x \div y)\), so that \(\min(x, y)\) is equal to the lesser of \(x\) and \(y\),

\[
\min(x, y) + [(x \div y) + (y \div x)]\{1 \div (y \div x)\} = x,
\]

\[
\min(x, y) + [(x \div y) + (y \div x)]\{1 \div (1 \div (y \div x))\} = y.
\]
Next we consider the particular case of (9) with \( n = 2 \), namely

\[
(9.2) \quad (x - y) + (y - z) + (z - x) = a.
\]

Let \([x, y, z]\) denote the left hand side of (9.2), let

\[
\min(x, y, z) = x - \{x - (y - (y - z))\},
\]

so that \(\min(x, y, z)\) is equal to the least of \(x, y, z\), let

\[
\max(x, y) = x + (y - x),
\]

the greater of \(x, y\), and finally let

\[
\mu(x, y, z) = 1 - [1 - \{x - \min(y, z)\}]
\]

so that \(\mu(x, y, z) = 0\) if \(x \leq y\) and \(x \leq z\) and \(\mu(x, y, z) = 1\) otherwise. Then

the general solution of (9.2) is

\[
(9.2^*) \quad \begin{align*}
x &= t + [a \mu(u, v, w) - \{\max(v, w) - u\}], \\
y &= t + [a \mu(v, w, u) - \{\max(w, u) - v\}], \\
z &= t + [a \mu(w, u, v) - \{\max(u, v) - w\}],
\end{align*}
\]

\(u, v, w\) not all equal.

To show that (9.2\(^*\)) satisfy 9.2 we consider in turn the six cases

\((\alpha)\) \( u < v \leq w \), \( (\beta) \) \( u \leq w < v \), \( (\gamma) \) \( v < w \leq u \), \( (\delta) \) \( v \leq u < w \),

\((\epsilon)\) \( w < u \leq v \), \( (\xi) \) \( w \leq v < u \).

In the first case \(\mu(u, v, w) = 0\), \(\mu(v, w, u) = \mu(w, u, v) = 1\) so that \(x = t\), \(y = t + \{a - (w - v)\}\), \(z = t + a\) and therefore (9.2), with the same result in the remaining cases. That (9.2\(^*\)) is the general solution follows from the identity

\[
(9.3) \quad x = \min(x, y, z) + \{[x, y, z] \mu(x, y, z) - (\max(y, z) - x)\}
\]

and the two corresponding results obtained by cyclic interchange of \(x, y, z\). Equation (9.3) shows that if \((x, y, z)\) is any solution of (9.2) then the value of \(x\) may be obtained from (9.2\(^*\)) by giving \(t\) the value \(\min(x, y, z)\) and \(u, v, w\) the values \(x, y, z\) respectively.

To prove (9.3) we consider six cases of which

\[
y \leq x \leq z, \quad y \neq z,
\]

is typical; in this case the right hand side of (9.3) becomes

\[
y + \{[x, y, z] - (x - x)\} = y + \{(z - y) - (z - x)\}
\]

\[
= y + (x - y) = x.
\]

The proofs in the remaining cases are similar (or simpler).
Finally, we remark that the general solution of (9) is expressible in the form

\[(9^*) \quad x_i = t + \left[ a \mu_i(u_r, n) - \max_i \left( \max_i(u_r, n) - u_i \right) \right], \quad 0 \leq i \leq n, \]

where not all \(u_r\) are equal, \(0 \leq r \leq n,\)

\[\mu_i(x_r, n) = 1 - \left( 1 - (x_i - \min_i(x_r, n)) \right),\]
\[\min(x_r, 0) = x_0, \quad \min(x_r, n + 1) = x_{n+1} - (x_{n+1} - \min_i(x_r, n)),\]
\[\max(x_r, 0) = x_0, \quad \max(x_r, n + 1) = x_{n+1} + (\max_i(x_r, n) - x_{n+1}),\]
\[x_r^i = x_r \quad \text{if} \ r < i, \quad x_r^i = x_{r+1} \quad \text{if} \ r \geq i,\]

and

\[\max_i(x_r, n) = x_0, \quad \max_i(x_r, n + 1) = \max_i(x_r^i, n),\]

so that \(\max_i(x_r, n)\) is the greatest value of \(x_r, r \neq i, \ 0 \leq r \leq n.\) The proof follows the same lines as that of (9.2*), the part played by (9.3) being taken by the identity

\[x_i = \min(x_r, n) + \left[ [x_r, n] \mu_i(x_r, n) - \left( \max_i(x_r, n) - x_i \right) \right], \quad 0 \leq i \leq n,\]

where \([x_r, n]\) denotes the left hand side of equation (9).