MATH. SCAND. 14 (1964), 106–110

A BOUNDARY PROBLEM FOR ANALYTIC LINEAR SYSTEMS WITH DATA ON INTERSECTING HYPERPLANES

JAN PERSSON

1. Introduction.

Let $x=x_1,\ldots,x_n$ be coordinates in C^n and put $D=\partial/\partial x$, $x^\alpha=x_1^{\alpha_1}\ldots x_n^{\alpha_n}$, $D^\alpha=(\partial/\partial x_1)^{\alpha_1}\ldots(\partial/\partial x_n)^{\alpha_n}$, $|\alpha|=\alpha_1+\ldots+\alpha_n$. When

$$P = P(x,D) = \sum a_\alpha(x) D^\alpha$$

is a differential operator we say that α is an index of P if $a_\alpha(x)\neq 0$. The set of indices of P will be denoted by (P). The order of P is the maximum of $|\alpha|$ as α ranges over (P). When f is analytic, we write

$$f = O(x^\alpha)$$

when $f(x)=x^\alpha g(x)$ with $g(x)$ analytic. An equivalent condition is that

$$x_k=0,\ j<\alpha_k \ \Rightarrow\ D_k^j f=0\ .$$

Notice that if f is analytic at the origin, the equation

$$D^\alpha u = f, \qquad u = O(x^\alpha)$$

has a unique solution holomorphic at the origin.

We say that an index β dominates a set of indices S if there exists a vector $a=(a_1,\ldots,a_n)$ with non-negative components a_j such that

$$\alpha\in S \ \Rightarrow\ a\alpha<a\beta\ .$$

Here $a\alpha$ is the scalar product $a_1\alpha_1+\ldots+a_n\alpha_n$. The geometrical meaning of this is that β is outside the convex hull of all γ such that $\gamma_1\leqq\alpha_1,\ldots,\gamma_n\leqq\alpha_n$ for some α in S. When $\beta^{(i)}$ dominates S_i for $i=1,\ldots,l$ we say that the domination is uniform if

$$\alpha\in S_i \ \Rightarrow\ a\alpha<a\beta^{(i)}$$

is true for all i and a vector a with non-negative components which is independent of i.

Received February 20, 1964.

THEOREM. *Let*

$$(1) \qquad D^{\beta(i)}u_i(x) = \sum_1^l P_{ij}(x,D)u_j(x) + f_i(x), \qquad i = 1,\ldots,l\,,$$

be a system of partial differential equations near the origin with the boundary conditions

$$(2) \qquad u_i - g_i = O(x^{\beta(i)}), \qquad i = 1,\ldots,l\,.$$

Let P_{ij}, f_i and g_i be analytic at the origin. Assume further that

$$(3) \qquad \beta^{(i)} \text{ dominates } (P_{i1}) \cup \ldots \cup (P_{il})$$

uniformly and that there exist integers $m_1,\ldots,m_l$ such that

$$(4) \qquad \text{order } P_{ij} \leqq m_i - m_j + |\beta^{(i)}|$$

for all i,j. Then the boundary problem (1), (2) *has a unique holomorphic solution u in a neighbourhood of the origin.*

The condition (4) is fully relevant only when $l > 1$. If $l = 1$, then (1), (2), (3) and (4) reduce to

$$(5) \qquad D^{\beta}u = P(x,D)u + f, \qquad u - g = O(x^{\beta}), \qquad \text{order } P \leqq |\beta|\,,$$

$$(6) \qquad \beta \text{ dominates } (P)\,.$$

The last condition is fulfilled if, e.g.,

$$(7) \qquad \alpha \in (P) \Rightarrow \alpha_n < \beta_n\,,$$

because then we can take $a = (0,\ldots,1)$. Under the assumptions (5) and (7), the theorem is due to Hörmander [4] and generalizes older results of Darboux, Goursat and Beudon (see Hadamard [3]). That (7) can be replaced by (6) was also observed by Hörmander (oral communication) and I thank him for his permission to publish the proof. For systems, i.e. when $l > 1$, the theorem is new. The special case when all $D^{\beta(i)}$ equal D_1 and no P contains indices α with $\alpha_1 \neq 0$ is due to Gårding, Kotake and Leray [2]. Another special case, extendable to non-linear systems, is due to Friedman [1]. — Our proof is a combination of the proof of Hörmander and that of Gårding, Leray and Kotake.

2. Proof of the theorem.

Putting $v = u - g$ we see that it suffices to consider the boundary problem

$$(1) \qquad D^{\beta(i)}v_i = \sum P_{ij}(x,D)v_j + f_i, \qquad v_i = O(x^{\beta(i)})\,.$$

To prove that (1) has a unique analytic solution we proceed in two steps. First, following Hörmander, we use (1.3) to make the coefficients of the

$$P_{ij} = \sum a_{ij\alpha}(x)D^{\alpha}$$

small by a dilatation of the coordinates.

We know that there exists an $a=(a_1,\ldots,a_n)$ with non-negative components such that

$$\alpha \in (P_{ij}) \Rightarrow a\beta^{(i)} > a\alpha .$$

Put

$$x_j' = e^{\tau a_j}x_j, \qquad \tau > 0 .$$

Then, if the coefficients and the f_i are analytic for

$$|x| = |x_1| + \ldots + |x_n| \leqq R < 1$$

the new coefficients and the new f_i will be analytic for $\sum e^{-\tau a_j}|x_j'| \leqq R$ and hence also for $|x'| \leqq R$. Further, the new coefficients are obtained from the old ones by the formula

$$a'_{ij\alpha}(x') = e^{-\tau(\beta^{(i)}-\alpha)a}a_{ij\alpha}(x)$$

and tend to zero uniformly when $\tau \to \infty$. Hence it suffices to prove the theorem when the coefficients of the P_{ij} are small.

To proceed further, we shall use an elementary lemma stated by Hörmander [4] with a slightly different notation.

LEMMA. *If* $v(x)$ *is analytic for* $|x| < R < 1$, *if* $a > 0$, $C > 0$ *and*

$$|D_k v(x)| \leqq C(R-|x|)^{-a-1}, \qquad x_k = 0 \Rightarrow v(x) = 0 ,$$

then

$$|v(x)| \leqq Ca^{-1}(R-|x|)^{-a-1} .$$

Conversely, if

$$|v(x)| \leqq C(R-|x|)^{-a}$$

then

$$|D_k v(x)| \leqq Ce(a+1)(R-|x|)^{-a-1} .$$

We shall solve (1) by successive approximations putting

$$D^{\beta^{(i)}}v_{i,r} = \sum P_{ij}v_{i,r-1} + f, \qquad v_{i,r} = O(x^{\beta^{(i)}}) , \tag{2}$$

where $v_{i,0}=0$. It is clear that this determines the $v_{i,r}$ uniquely. The functions

$$w_{i,r} = v_{i,r} - v_{i,r-1}, \qquad r > 0$$

satisfy the same recursion formulas with $f=0$. We are going to show that there exist constants c and C such that

$$(3) \qquad |w_{i,r}| \leqq Cc^{r-1}r^{m_i}d(x)^{-(rs+m_i)}$$

for all r. Here $|x| < R$, $s_i = |\beta^{(i)}|$, $s = \max s_i$ ($\geqq 1$) and $d(x) = (R - |x|)$, and we have assumed that all m_i are positive, which is no restriction. In fact, (3) holds for $r = 1$ if C is large enough. Assume that it holds for r. Then, by the second part of the lemma, we get

$$|D^\alpha w_{j,r}| \leqq Cc^{r-1}r^{m_j}e^{|\alpha|}(rs + m_j + |\alpha|)^{|\alpha|}\, d(x)^{-(rs+m_j+|\alpha|)} .$$

Now for $\alpha \in (P_{ij})$ we have, by (1.4),

$$|\alpha| + m_j \leqq m_i + s_i \leqq m_i + s \leqq M + s$$

where $M = \max m_i$, so that, in this case

$$|D^\alpha w_{j,r}| \leqq Cc^{r-1}r^{m_j}e^{s+M}\big((r+1)s + M\big)^{m_i - m_j + s_i} d(x)^{-((r+1)s+m_i)} .$$

Since

$$\begin{aligned} r^{m_j}\big((r+1)s + M\big)^{m_i-m_j+s_i} &= r^{m_i+s_i}(s + s/r + M/r)^{m_i+s_i}(s + s/r + M/r)^{-m_j} \\ &\leqq (2s + M)^{s+M} r^{m_i+s_i} \end{aligned}$$

for all $r \geqq 1$, putting

$$c_0 = 2s + M ,$$

we also get

$$|D^\alpha w_{j,r}| \leqq Cc^{r-1}(c_0 e)^{s+M} r^{m_i+s_i}\, d(x)^{-((r+1)s+m_i)} .$$

Now let

$$\gamma = \max_{|x|<R} \max_i \sum_j \sum_\alpha |a_{ij\alpha}(x)| .$$

Then, since (2) holds for w with $f = 0$, we get

$$|D^{\beta^{(i)}} w_{i,r+1}| \leqq Cc^{r-1}\gamma(c_0 e)^{s+M} r^{m_i+s_i}\, d(x)^{-((r+1)s+m_i)}$$

so that by virtue of the first part of the lemma

$$\begin{aligned} |w_{i,r+1}| &< Cc^{r-1}\gamma(c_0e)^{s+M} r^{m_i+s_i}\big((r+1)s + m_i - 1\big)^{-s_i}\, d(x)^{-((r+1)s+m_i)} \\ &\leqq Cc^{r-1}\gamma(c_0e)^{s+M}(r+1)^{m_i}\, d(x)^{-((r+1)s+m_i)} . \end{aligned}$$

Hence, if

$$c = \gamma(c_0 e)^{s+M} ,$$

(3) holds for all r. Now γ can be chosen arbitrarily small and if we make it so small that $c < R^s$ then $c/d(x) < 1$ for small x so that the series $\sum_{n=1}^{\infty} w_{i,r}(x)$ converges to a holomorphic solution of (1) in a neighbourhood of the origin.

If v satisfies (1) with $f = 0$ then $w_r = v$, $r = 1, 2, \ldots$, satisfies the recursive formula (2) with $f = 0$ and hence by what we have proved $v = w_r$ tends to zero if $r \to \infty$ and γ is small enough. Hence v vanishes. The proof if complete.

REFERENCES

1. A. Friedman, *A new proof and generalizations of the Cauchy–Kovalewski theorem*, Trans. Amer. Math. Soc. 98 (1961), 1–20.
2. L. Gårding, T. Kotake and J. Leray, *Uniformisation et développement asymptotique de la solution du problème de Cauchy linéaire à données holomorphes* (problème de Cauchy I$^{\text{bis}}$ et VI). To be published in Journal de mathématiques pures et appliquées.
3. J. Hadamard, *Le problème de Cauchy et les équations aux derivées partielles linéaire hyperboliques*, Paris, 1932.
4. L. Hörmander, *Linear partial differential operators*, Berlin, 1963.

UNIVERSITY OF LUND, SWEDEN