A BOUNDARY PROBLEM
FOR ANALYTIC LINEAR SYSTEMS WITH DATA ON
INTERSECTING HYPERPLANES

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1. Introduction.

Let $x = x_1, \ldots, x_n$ be coordinates in $C^n$ and put $D = \partial/\partial x$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. When

$$P = P(x,D) = \sum a_\alpha(x) D^\alpha$$

is a differential operator we say that $\alpha$ is an index of $P$ if $a_\alpha(x) \neq 0$. The set of indices of $P$ will be denoted by $(P)$. The order of $P$ is the maximum of $|\alpha|$ as $\alpha$ ranges over $(P)$. When $f$ is analytic, we write

$$f = O(x^\alpha)$$

when $f(x) = x^\alpha g(x)$ with $g(x)$ analytic. An equivalent condition is that

$$x_k = 0, j < \alpha_k \Rightarrow D_k f = 0.$$ 

Notice that if $f$ is analytic at the origin, the equation

$$D^\alpha u = f, \quad u = O(x^\alpha)$$

has a unique solution holomorphic at the origin.

We say that an index $\beta$ dominates a set of indices $S$ if there exists a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ with non-negative components $\alpha_j$ such that

$$\alpha \in S \Rightarrow a\alpha < a\beta.$$ 

Here $a\alpha$ is the scalar product $a_1\alpha_1 + \cdots + a_n\alpha_n$. The geometrical meaning of this is that $\beta$ is outside the convex hull of all $\gamma$ such that $\gamma_1 \leq \alpha_1, \ldots, \gamma_n \leq \alpha_n$ for some $\alpha$ in $S$. When $\beta^{(i)}$ dominates $S_i$ for $i = 1, \ldots, l$ we say that the domination is uniform if

$$\alpha \in S_i \Rightarrow a\alpha < a\beta^{(i)}$$

is true for all $i$ and a vector $a$ with non-negative components which is independent of $i$.

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Theorem. Let
\[ D^{\beta(i)} u_i(x) = \sum_{j=1}^{l} P_{ij}(x, D)u_j(x) + f_i(x), \quad i = 1, \ldots, l, \]
be a system of partial differential equations near the origin with the boundary conditions
\[ u_i - g_i = O(x^{\beta(i)}), \quad i = 1, \ldots, l. \]
Let $P_{ij}, f_i$ and $g_i$ be analytic at the origin. Assume further that
\[ \beta^{(i)} \text{ dominates } (P_{i1}) \cup \ldots \cup (P_{il}) \]
uniformly and that there exist integers $m_1, \ldots, m_l$ such that
\[ \text{order } P_{ij} \leq m_i - m_j + |\beta^{(i)}| \]
for all $i,j$. Then the boundary problem (1), (2) has a unique holomorphic solution $u$ in a neighbourhood of the origin.

The condition (4) is fully relevant only when $l > 1$. If $l = 1$, then (1), (2), (3) and (4) reduce to
\[ D^\beta u = P(x, D)u + f, \quad u - g = O(x^\beta), \quad \text{order } P \leq |\beta|, \]
\[ \beta \text{ dominates } (P). \]
The last condition is fulfilled if, e.g.,
\[ \alpha \in (P) \Rightarrow \alpha_n < \beta_n, \]
because then we can take $a = (0, \ldots, 1)$. Under the assumptions (5) and (7), the theorem is due to Hörmander [4] and generalizes older results of Darboux, Goursat and Beudon (see Hadamard [3]). That (7) can be replaced by (6) was also observed by Hörmander (oral communication) and I thank him for his permission to publish the proof. For systems, i.e. when $l > 1$, the theorem is new. The special case when all $D^{\beta(i)}$ equal $D_1$ and no $P$ contains indices $\alpha$ with $\alpha_1 \neq 0$ is due to Gårding, Kotake and Leray [2]. Another special case, extendable to non-linear systems, is due to Friedman [1]. — Our proof is a combination of the proof of Hörmander and that of Gårding, Leray and Kotake.

2. Proof of the theorem.

Putting $v = u - g$ we see that it suffices to consider the boundary problem
\[ D^{\beta(i)} v_i = \sum P_{ij}(x, D)v_j + f_i, \quad v_i = O(x^{\beta(i)}). \]
To prove that (1) has a unique analytic solution we proceed in two steps. First, following Hörmander, we use (1.3) to make the coefficients of the
\[ P_{ij} = \sum a_{ij\alpha}(x)D^\alpha \]
small by a dilatation of the coordinates.

We know that there exists an \( a = (a_1, \ldots, a_n) \) with non-negative components such that
\[ \alpha \in (P_{ij}) \Rightarrow a_\beta(i) > a\alpha . \]
Put
\[ x_j' = e^{r\alpha_j}x_j, \quad r > 0 . \]
Then, if the coefficients and the \( f_i \) are analytic for
\[ |x| = |x_1| + \ldots + |x_n| \leq R < 1 \]
the new coefficients and the new \( f_i \) will be analytic for \( \sum e^{-r\alpha_j}|x_j'| \leq R \) and hence also for \( |x'| \leq R \). Further, the new coefficients are obtained from the old ones by the formula
\[ a_{ij\alpha}'(x') = e^{-r(\beta(i) - \alpha)\alpha}a_{ij\alpha}(x) \]
and tend to zero uniformly when \( r \to \infty \). Hence it suffices to prove the theorem when the coefficients of the \( P_{ij} \) are small.

To proceed further, we shall use an elementary lemma stated by Hörmander [4] with a slightly different notation.

**Lemma.** If \( v(x) \) is analytic for \( |x| < R < 1 \), if \( a > 0 \), \( C > 0 \) and
\[ |D_kv(x)| \leq C(R - |x|)^{-a-1}, \quad x_k = 0 \Rightarrow v(x) = 0 , \]
then
\[ |v(x)| \leq Ca^{-1}(R - |x|)^{-a-1} . \]
Conversely, if
\[ |v(x)| \leq C(R - |x|)^{-a} \]
then
\[ |D_kv(x)| \leq Ce(a + 1)(R - |x|)^{-a-1} . \]

We shall solve (1) by successive approximations putting
\[ D^{\beta(i)}v_{i,r} = \sum P_{ij}v_{i,j-1} + f, \quad v_{i,0} = O(x^{\beta(i)}) , \]
where \( v_{i,0} = 0 \). It is clear that this determines the \( v_{i,r} \) uniquely. The functions
\[ w_{i,r} = v_{i,r} - v_{i,r-1}, \quad r > 0 \]
satisfy the same recursion formulas with \( f = 0 \). We are going to show that there exist constants \( c \) and \( C \) such that
(3) \[ |w_{x,r}| \leq Cc^{r-1}r^{m_i}d(x)^{-(r+s+m_i)} \]

for all \( r \). Here \( |x| < R \), \( s_i = |\beta^{(i)}| \), \( s = \max s_i \ (\geq 1) \) and \( d(x) = (R - |x|) \), and we have assumed that all \( m_i \) are positive, which is no restriction. In fact, (3) holds for \( r = 1 \) if \( C \) is large enough. Assume that it holds for \( r \). Then, by the second part of the lemma, we get

\[ |D^\alpha w_{x,r}| \leq Cc^{r-1}r^{m_i}e^{s+M}(rs + m_i + |x|)^{s+M}d(x)^{-(r+s+m_i+|x|)} \]

Now for \( \alpha \in (P_i) \) we have, by (1.4),

\[ |\alpha| + m_j \leq m_i + s_i \leq m_i + s \leq M + s \]

where \( M = \max m_i \), so that, in this case

\[ |D^\alpha w_{x,r}| \leq Cc^{r-1}r^{m_i}e^{s+M}(r+1)s + M)^{m_i}d(x)^{-(r+1)s+m_i} \]

Since

\[ r^{m_i}(r+1)s + M)^{m_i} \leq r^{m_i+s_i}(s + s/r + M/r)^{m_i+s_i} \leq (2s + M)^{s+M}r^{m_i+s_i} \]

for all \( r \geq 1 \), putting

\[ c_0 = 2s + M \]

we also get

\[ |D^\alpha w_{x,r}| \leq Cc^{r-1}(c_0e)^{s+M}r^{m_i+s_i}d(x)^{-(r+1)s+m_i} \]

Now let

\[ \gamma = \max_{|x| < R} \max_i \sum_j \sum_{\alpha} |a_{ij\alpha}(x)| \]

Then, since (2) holds for \( w \) with \( f = 0 \), we get

\[ |D^\alpha w_{x,r+1}| \leq Cc^{r-1}\gamma(c_0e)^{s+M}r^{m_i+s_i}d(x)^{-(r+1)s+m_i} \]

so that by virtue of the first part of the lemma

\[ |w_{x,r+1}| \leq Cc^{r-1}\gamma(c_0e)^{s+M}r^{m_i+s_i}(r+1)s + m_i - 1)^{-s_i}d(x)^{-(r+1)s+m_i} \leq Cc^{r-1}\gamma(c_0e)^{s+M}(r+1)^{m_i}d(x)^{-(r+1)s+m_i} \]

Hence, if

\[ c = \gamma(c_0e)^{s+M} \]

(3) holds for all \( r \). Now \( \gamma \) can be chosen arbitrarily small and if we make it so small that \( c < R^s \) then \( c/d(x) < 1 \) for small \( x \) so that the series \( \sum_{n=1}^\infty w_{x,r}(x) \) converges to a holomorphic solution of (1) in a neighbourhood of the origin.

If \( v \) satisfies (1) with \( f = 0 \) then \( w_r = v, r = 1, 2, \ldots \), satisfies the recursive formula (2) with \( f = 0 \) and hence by what we have proved \( v = w_r \) tends to zero if \( r \to \infty \) and \( \gamma \) is small enough. Hence \( v \) vanishes. The proof if complete.
REFERENCES


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