STABILITY OF LINEAR DIFFERENTIAL EQUATIONS
IN BANACH ALGEBRAS

GERT ALMKVIST

Introduction.
We consider a Banach algebra \( A \) over the real or complex numbers with identity element (denoted \( e \)). All norms, \( N(\cdot) \), are assumed to be natural, that is \( N(xy) \leq N(x)N(y) \) and \( N(e) = 1 \). We study differential equations of the type

\[
(1) \quad u'(t) = a(t)u(t) \quad \text{where} \quad u(0) = e.
\]

Here \( u \) and \( a \) are functions from the positive reals \( R^+ \) into \( A \). \( u'(t) \) is defined by the relation

\[
N\left(\frac{u(t+h) - u(t)}{h} - u'(t)\right) \to 0 \quad \text{as} \quad h \to 0.
\]

We always assume that \( a(t) \) is Bochner integrable (see [2, p. 79]) over any finite interval of \( R^+ \). This implies that there exists a unique solution of (1) (see [6, p. 521]). The differential equation (1) is said to be stable if there exists a constant \( M \) such that

\[
N(u(t)) \leq M \quad \text{for all} \quad t \geq 0.
\]

It is clear that stability is a topological property, i.e. it is preserved if we introduce an equivalent norm of \( A \). It is therefore natural to try to find topological conditions on \( a(t) \) making (1) stable.

The first part considers the case when \( a(t) \) is constant. Here we can solve (1): \( u(t) = \exp(ta) \). The spectrum \( \sigma(a) \) of \( a \), is independent of the norm and we find some sufficient conditions for stability, generalizing well known theorems for matrix algebras.

We introduce the Gâteaux differential belonging to a norm \( N \)

\[
\Phi_N(a) = \lim_{\alpha \to +0} \frac{N(e + \alpha a) - 1}{\alpha}.
\]

A technical condition for stability is obtained:

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Theorem 3. The function $\exp(ta)$ is bounded for $t \geq 0$ if and only if there exists a norm $\Phi$ such that $\Phi_N(a) \leq 0$.

In the second part, $a(t)$ is not necessarily constant. We prove

Theorem 4. $N(u(t)) \leq \exp \left( \int_0^t \Phi_N(a(s)) \right) ds$.

From this theorem we get some corollaries concerning stability of (1).

1. $a(t)$ constant.

We study relations between the situation of the spectrum $\sigma(a)$ in the complex plane and stability of (1). If $\sigma(a)$ contains a point $\lambda_0$ with $\Re \lambda_0 > 0$ then

$$N(\exp(ta)) \geq \nu(\exp(ta)) \geq \exp(t \Re \lambda_0),$$

where $\nu(x) = \sup \{|\lambda| \mid \lambda \in \sigma(x)\}$ is the spectral radius of $x$. Hence

$$\sigma(a) \subseteq \{\lambda \mid \Re \lambda \leq 0\}$$

is necessary for stability. We also have the following sufficient condition.

Theorem 1. If $\sigma(a) \subseteq \{\lambda \mid \Re \lambda < 0\}$ then $\exp(ta) \to 0$ when $t \to \infty$.

Proof. Put $b = \exp(a)$. There exists a $\delta > 0$ such that

$$\sigma(b) \subseteq \{\lambda \mid |\lambda| \leq \exp(-\delta)\}.$$

We get

$$\nu(b) = \lim_{n \to \infty} (N(b^n))^{1/n} = \sup_{\lambda \in \sigma(b)} |\lambda| \leq e^{-\delta} < 1.$$

It follows that $N(b^n) \to 0$ as $n \to \infty$ and this implies $\exp(ta) \to 0$ as $t \to \infty$.

If $\sigma(a)$ contains points on the imaginary axis the problem of finding a sufficient condition of stability is more difficult. Theorem 2 is a generalization of a well known theorem for matrix algebras.

Theorem 2. Let $A = B(X)$ be the algebra of bounded linear operators from a Banach space $X$ into itself.

Let further $a = b + c$ be a spectral operator in $A$ with resolution $E(A)$ of the identity, $b$ the scalar and $c$ the quasinilpotent part of $a$. If

$$\sigma(a) \subseteq \{\lambda \mid \Re \lambda \leq 0\},$$

and

$$\Lambda_1 = \sigma(A) \cap \{\lambda \mid \Re \lambda = 0\}$$

is both open and closed as a subset of $\sigma(A)$ and $c E(\Lambda_1) = 0$, then $\exp(ta)$ is bounded for $t \geq 0$. 
Proof. The set \( \Delta_2 = \sigma(A) - \Delta_1 \) is a closed subset of \( \{ \lambda \mid \text{Re}\lambda < 0 \} \). We have a decomposition \( a = a_1 + a_2 \) where \( a_k = a \mid E(A_k)X, \ k = 1, 2 \). Since \( a_1 \) is scalar we get from the definition of spectral operator (cf. [2])

\[
\sigma(a_1) \subset \overline{\Delta}_1 = \Delta_1 \subset \{ \lambda \mid \text{Re}\lambda = 0 \},
\]

\[
\sigma(a_2) \subset \overline{\Delta}_2 = \Delta_2 \subset \{ \lambda \mid \text{Re}\lambda < 0 \},
\]

and

\[
N(\exp(ta_1)) \leq \text{const} \cdot \sup_{\lambda \in \sigma(a_1)} |\exp(t\lambda)| = \text{const}.
\]

Theorem 1 implies that \( \exp(ta_2) \) is bounded and this finishes the proof.

We now consider the Gateau differential

\[
\Phi_N(a) = \lim_{\alpha \to +0} \frac{N(e + \alpha a) - 1}{\alpha}.
\]

We write down the following simple consequences of the definition (see [4, p. 25]):

\[
|\Phi_N(a)| \leq N(a),
\]

\[
\Phi_N(a + b) \leq \Phi_N(a) + \Phi_N(b),
\]

\[
\Phi_N(a) = \inf_{\alpha > 0} \frac{N(e + \alpha a) - 1}{\alpha} = \lim_{\alpha \to +0} \frac{\log N(\exp(\alpha a))}{\alpha}.
\]

The next theorem is a precise characterization of those constants \( a \) for which (1) is stable.

**Theorem 3.** The function \( \exp(ta) \) is bounded for \( t \geq 0 \) if and only if there exists a norm \( N \) for which \( \Phi_N(a) \leq 0 \).

**Proof.** Assume that \( N(\exp(ta)) \leq K \) for all \( t \geq 0 \). By a construction due to L. Ingelstam (see [4, p. 26]) it is possible to find an equivalent norm \( N_2 \) such that \( N_2(\exp(ta)) \leq 1 \) for all \( t \geq 0 \). Put

\[
N_1(x) = \sup_{t \geq 0} N(\exp(ta) \cdot x).
\]

Then

\[
N(x) \leq N_1(x) \leq K \cdot N(x) \quad \text{for all } x,
\]

that is, \( N \) and \( N_1 \) are equivalent, but \( N_1(1) \neq 1 \) in general. We therefore introduce

\[
N_2(x) = \sup_{y > 0} \frac{N_1(xy)}{N_1(y)}.
\]

It is easily seen that \( N_2 \) is natural, equivalent to \( N \) and

\[
N_2(\exp(ta)) \leq 1 \quad \text{for all } t \geq 0.
\]
From (5) we then get
\[ \Phi_{N_2}(a) = \lim_{t \to +0} \frac{\log N_2(\exp ta)}{t} \leq 0 , \]
which proves the necessity.

If \( \Phi_{N}(a) \leq 0 \) then Corollary 1 of Theorem 4 (see below) implies that \( N(\exp(ta)) \leq 1 \) for all \( t \geq 0 \), so the condition is sufficient.

\[ \textbf{2. a(t) not necessarily constant.} \]

We first observe that \( a(t) \) Bohner integrable over any finite interval implies that \( \Phi(a(t)) \) is integrable over the same type of intervals. This follows from the fact that \( \Phi_N(a(t)) \leq N(a(t)) \) where \( N(a(t)) \) is integrable and that \( \Phi_N(a(t)) \) is an infimum of a sequence of measurable functions (5).

**Theorem 4.** \( N(u(t)) \leq \exp \int_0^t \Phi_N(a(s)) \, ds. \)

**Proof.** For a fixed \( t > 0 \) we construct the solution by successive approximations. Consider a partition
\[ 0 = t_0 < t_1 < \ldots < t_n = t . \]
We define the following step functions:
\[ b_n(s) = a(\xi_k), \quad g_n(s) = \Phi(a(\xi_k)), \]
\[ v_n(s) = \exp((s-t_k)a(\xi_k)) \cdot \exp((t_k-t_{k-1})a(\xi_{k-1})) \ldots \exp(t_1a(\xi_0)) \]
for \( s \) satisfying \( t_k \leq s < t_{k+1} \). Here \( t_k \leq \xi_k < t_{k+1} \). Integrating \( v(s) \) we get
\[ v_n(t) = e + \int_0^t b_n(s) v_n(s) \, ds . \]
We have a similar equation for \( u(t) \) and using a well known lemma (see [1, p. 35]) we find
\[ N(u(t) - v_n(t)) \leq \sup_{0 \leq s \leq t} N(u(s)) \cdot \int_0^t N(a(s) - b_n(s)) \, ds \cdot \exp \int_0^t N(b_n(s)) \, ds . \]
By choosing a convenient partition we can make
\[ \left| \int_0^t (\Phi(a(s)) - g_n(s)) \, ds \right| \quad \text{and} \quad \int_0^t N(a(s) - b_n(s)) \, ds , \]
and hence \( N(u(t) - v_n(t)) \), as small as we please. From (6) we get
\[ \log N(v_n(t)) \leq \sum_{k=0}^{n-1} \log N\left(\exp\left((t_{k+1} - t_k)a(\xi_k)\right)\right) \frac{t_{k+1} - t_k}{t_{k+1} - t_k} \cdot \]

If necessary dividing the intervals \([t_k, t_{k+1})\) further (leaving \(a(\xi_k)\) unchanged in the whole interval \([t_k, t_{k+1})\)) and using (5) we can also obtain

\[
\log N(v_n(t)) - \int_0^t g_n(s) \, ds < \varepsilon \quad \text{for an arbitrary } \varepsilon > 0.
\]

From these facts it follows that

\[
N(u(t)) \leq \exp \int_0^t \Phi_n(a(s)) \, ds,
\]

which ends the proof.

**Corollary 1.** If there exists a norm \(N\) such that

\[
\limsup_{t \to \infty} \int_0^t \Phi_N(a(s)) \, ds < \infty,
\]

then \(u' = a(t)u\) is stable.

From (3) we get

**Corollary 2.** If \(\int_0^\infty N(a(t)) \, dt < \infty\) for some norm \(N\) then \(u' = a(t)u\) is stable.

**Corollary 3.** Let \(A\) be a C\(^*\)-algebra. If \(a(t)\) is normal \((aa^* = a^*a)\) for every \(t\) and

\[
\sigma(a(t)) \subset \{\lambda \mid \text{Re}\lambda \leq f(t)\} \quad \text{where} \quad \limsup_{t \to \infty} \int_0^t f(s) \, ds < \infty,
\]

then \(u' = a(t)u\) is stable.

**Proof.** If \(a\) is normal then \(\exp(\alpha a)\) is also normal and we get

\[
N(\exp(\alpha a(t))) = v(\exp(\alpha a(t))) \leq \exp(\alpha f(t))
\]

for the C\(^*\)-norm \(N\) when \(\alpha > 0\). Applying (5) we have

\[
\Phi(a(t)) = \lim_{\alpha \to +0} \frac{\log N(\exp(\alpha a(t)))}{\alpha} \leq f(t)
\]

and

\[
\int_0^t \Phi_N(a(s)) \, ds \leq \int_0^t f(s) \, ds.
\]

Finally we use Corollary 1 and the proof is finished.

**Remark.** If \(a(t)\) is not normal the spectrum does not tell us much about the stability. It may for instance happen that \(\sigma(a(t)) \subset\)
\{ \lambda \mid \Re \lambda = -\frac{1}{2} \} \text{ for all } t \text{ and } u' = a(t)u \text{ has unbounded solutions (see [5, p. 310]).}

**Theorem 5.** Assume that \( u' = a(t)u \) is stable. If there exists a norm \( N \) such that

\[
\limsup_{t \to \infty} \int_{0}^{t} \Phi_{N}(-a(s))ds < \infty \quad \text{and} \quad \int_{0}^{\infty} N(b(t))dt < \infty \, ,
\]

then \( u' = (a(t) + b(t))u \) is stable.

**Proof.** Let \( u_{0}(t) \) be the bounded solution of \( u' = a(t)u_{0} \) with \( u_{0}(0) = e \) as usual. We have the formula

\[
(7) \quad u(t) = u_{0}(t) + \int_{0}^{t} u_{0}(t)u^{-1}(s)b(s)u(s)ds \, ,
\]

which one easily verifies by calculating the derivatives of both sides (for the existence of \( u_{0}^{-1}(s) \) see [6, p. 521]). Differentiating \( u_{0}^{-1} \) we get

\[
(u_{0}^{-1})' = u_{0}^{-1}(-a(t)) \, .
\]

The same proof as of Theorem 4 can be used to show that

\[
N(u_{0}^{-1}(t)) \leq \exp \int_{0}^{t} \Phi(-a(s))ds \leq C \, ,
\]

a constant independent of \( t \). From (7) we get (using the lemma again)

\[
N(u(t)) \leq N(u_{0}(t)) \cdot \exp \left\{ C \cdot \sup_{0 \leq s \leq t} N(u_{0}(s)) \cdot \int_{0}^{t} N(b(s))ds \right\} \, ,
\]

and from the conditions we deduce that \( u(t) \) is bounded.

**References**


The Royal Institute of Technology, Stockholm, Sweden