APPROXIMATION
THEOREMS OF BOREL AND FUJIWARA

ASMUS L. SCHMIDT

1. Introduction.

The purpose of this note is to give a unified proof of one theorem of Borel [1] and two theorems of Fujiwara [2] [3], stated as theorem I, II, III, respectively, below. However, I believe that the part of theorem III dealing with the approximation of a rational number $\xi$ is new. The method of proof is based on an idea of Khintchine [4].

For any real number $\xi$ let $[a_0, a_1, \ldots]$ be the regular continued fraction expansion (if $\xi$ is rational, we consider any of the two continued fraction expansions), and let $p_0/q_0, p_1/q_1, \ldots$ be the corresponding convergents. The properties of continued fractions to be used are

1. $a_0$ is an integer; $a_n$ is a positive integer, $n \geq 1$.
2. $q_{n+1} = a_{n+1}q_n + q_{n-1}, n \geq 1$, with $q_0 = 1, q_1 = a_1$.
3. $p_nq_{n-1} - p_{n-1}q_n = \pm 1, n \geq 1$.
4. $\xi$ lies between two consecutive convergents.

With this notation theorems I, II, III may be formulated as follows.

**Theorem I.** At least one of $p_{n-1}/q_{n-1}, p_n/q_n, p_{n+1}/q_{n+1}, n \geq 1$, satisfies the inequality

\[
|\xi - \frac{p}{q}| < \frac{1}{5^4 q^2}.
\]

**Theorem II.** If $a_{n+1} \geq 2, n \geq 1$, then at least one of $p_{n-1}/q_{n-1}, p_n/q_n, p_{n+1}/q_{n+1}$ satisfies the inequality

\[
|\xi - \frac{p}{q}| < \frac{1}{8^4 q^2}.
\]

**Theorem III.** If $a_{n+1} \geq 2, n \geq 1$, then either $p_n/q_n$ or both of $p_{n-1}/q_{n-1}, p_{n+1}/q_{n+1}$ satisfy the inequality

\[
|\xi - \frac{p}{q}| \leq \frac{1}{5^2 q^2}.
\]

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where the equality sign can only occur if \( \xi = a_0 + \frac{2}{n}, n = 1 \), or \( \xi = a_0 + \frac{3}{n}, n = 2 \), and for such \( \xi \) only if the shorter form of the two continued fraction expansions is considered.

2. Two lemmas.

**Lemma 1.** Let \( q, q' \) be positive integers. Then

\[
\frac{1}{qq'} < \frac{1}{K} \left( \frac{1}{q^2} + \frac{1}{q'^2} \right)
\]

whenever \( q'/q > f(K) \) or \( q'/q > f(K) \), where \( f(K) = \frac{1}{2}(K + (K^2 - 4)^{\frac{1}{2}}) \). In particular \( f(5^4) = \frac{1}{2}(5^4 + 1), f(8^4) = 2^i + 1, f(\frac{7}{2}) = 2 \).

**Proof.** (8) is equivalent to \((q'/q)^2 - Kq'q + 1 > 0\), which immediately yields the lemma.

**Lemma 2.** Let \( p/q \leq \xi \leq p'/q' \), where \( p, p', q, q' \) are integers with \( q, q' > 0 \) and \( p'q - pq' = 1 \). If either \( q'/q > f(K) \) or \( q'/q' > f(K) \), then either

\[
\left| \frac{\xi - p}{q} \right| < \frac{1}{Kq^2} \quad \text{or} \quad \left| \frac{\xi - p'}{q'} \right| < \frac{1}{Kq'^2}.
\]

**Proof.** By lemma 1, \( q, q' \) satisfy (8), i.e.

\[
\frac{p'}{q'} - \frac{p}{q} = \frac{1}{qq'} < \frac{1}{K} \left( \frac{1}{q^2} + \frac{1}{q'^2} \right),
\]

whence

\[
\frac{p}{q} + \frac{1}{Kq^2} > \frac{p'}{q'} - \frac{1}{Kq'^2}.
\]

This proves lemma 2.

3. Proof of theorems I, II, III.

1) If \( q_n/q_{n-1} > \frac{1}{2}(5^i + 1) \), either \( p_{n-1}/q_{n-1} \) or \( p_n/q_n \) satisfies (5) by (3), (4) and lemma 2 \((K = 5^i)\). If on the contrary \( q_n/q_{n-1} < \frac{1}{2}(5^i + 1) \), then \( q_{n-1}/q_n > \frac{1}{2}(5^i - 1) \). Hence \( q_{n+1}/q_n = a_{n+1} + q_{n-1}/q_n > 1 + \frac{1}{2}(5^i - 1) = \frac{1}{2}(5^i + 1) \) by (1), (2), consequently in this case either \( p_n/q_n \) or \( p_{n+1}/q_{n+1} \) satisfies (5). This proves theorem I.

2) If \( q_n/q_{n-1} > 2^i + 1 \), either \( p_{n-1}/q_{n-1} \) or \( p_n/q_n \) satisfies (6) by (3), (4) and lemma 2 \((K = 8^i)\). If on the contrary \( q_n/q_{n-1} < 2^i + 1 \), then \( q_{n-1}/q_n > 2^i - 1 \). Hence

\[
q_{n+1}/q_n = a_{n+1} + q_{n-1}/q_n > 2 + 2^i - 1 = 2^i + 1
\]
by (2) and the assumption \( a_{n+1} \geq 2 \) of theorem II, consequently in this case either \( p_n/q_n \) or \( p_{n+1}/q_{n+1} \) satisfies (6). This proves theorem II.

3) If \( p_n/q_n \) satisfies (7) with strict inequality, we are finished. On the contrary assume that

\[
\left| \frac{\xi - p_n}{q_n} \right| \geq \frac{1}{\frac{5}{2} q_n^2},
\]

then by (3), (4)

\[
\frac{1}{q_n q_{n+1}} = \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \left| \frac{\xi - p_n}{q_n} + \frac{\xi - p_{n+1}}{q_{n+1}} \right| \geq \frac{1}{\frac{5}{2} q_n^2}
\]

or \( q_{n+1}/q_n \leq \frac{5}{2} \), with strict inequality unless \( \xi = p_{n+1}/q_{n+1} \) and

\[
\left| \frac{\xi - p_n}{q_n} \right| = \frac{1}{\frac{5}{2} q_n^2}.
\]

Now \( q_{n+1}/q_n = a_{n+1} + q_{n-1}/q_n \), hence \( a_{n+1} = 2 \) and \( q_{n-1}/q_n \leq \frac{1}{2} \) with strict inequality unless \( \xi = p_{n+1}/q_{n+1} \) and \( q_{n+1}/q_n = \frac{5}{2} \).

In any case \( q_{n+1}/q_n > 2 \), so by lemma 2 (\( K = \frac{5}{2} \)) either \( p_n/q_n \) or \( p_{n+1}/q_{n+1} \) satisfies (7) with strict inequality, i.e. \( p_{n+1}/q_{n+1} \) does so, since \( p_n/q_n \) does not by assumption. If further \( q_{n-1}/q_n < \frac{1}{2} \), \( q_n/q_{n-1} > 2 \), so by lemma 2 (\( K = \frac{5}{2} \)) either \( p_{n-1}/q_{n-1} \) or \( p_n/q_n \) satisfies (7) with strict inequality, i.e. \( p_{n-1}/q_{n-1} \) does so, since \( p_n/q_n \) does not by assumption. This proves the main case of theorem III.

There remains only to show that \( \xi = p_{n+1}/q_{n+1}, q_{n+1}/q_n = \frac{5}{2}, a_{n+1} = 2, q_{n-1}/q_n = \frac{1}{2} \) leads to the exceptional case of theorem III.

By (3) \( q_{n-1}, q_n \) are relatively prime and by (1), (2) \( 1 = q_0 \leq a_1 = q_1 < q_2 < q_3 < \ldots \). This requires \( q_{n-1} = 1, q_n = 2, q_{n+1} = 5 \) and either \( n = 1 \) or \( n = 2 \) in which case \( a_1 = 1 \). Hence by (2) either \( \xi = [a_0, 2, 2] = a_0 + \frac{5}{6} \) or \( \xi = [a_0, 1, 1, 2] = a_0 + \frac{5}{6} \), where \( a_0 \) is an integer. In the first case

\[
\frac{p_0}{q_0} = a_0 + \frac{1}{1}, \quad \frac{p_1}{q_1} = \frac{2a_0 + 1}{2}
\]

both satisfy (7) with equality. Similarly with

\[
\frac{p_1}{q_1} = a_0 + \frac{1}{1}, \quad \frac{p_1}{q_2} = \frac{2a_0 + 1}{2}
\]

in the second case. This completes the proof of theorem III.

REFERENCES

1. É. Borel, Sur l’approximation des nombres irrationnels par des nombres rationnels,


UNIVERSITY OF COPENHAGEN, DENMARK