INARIANT UNIFORMITIES FOR COSET SPACES

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S. N. Hudson [2] and T. S. Wu [4] have shown that, if \( H \) is a compact subgroup of a topological group \( G \), then \( G/H \) has a right invariant uniformity (invariant under action of \( G \) on \( G/H \)) and that the uniformity is pseudometrizable if \( G/H \) satisfies the first axiom of countability. It is known that, if \( G \) is Hausdorff and satisfies the first axiom of countability and \( H \) is closed then \( G/H \) is metrizable (Montgomery and Zippin [3, p. 36]). In this paper we obtain a result which includes those of Hudson and Wu and covers some other cases not included in the theorem of Montgomery and Zippin.

For each symmetric neighborhood \( U \) of the identity \( e \) of \( G \) let

\[
U^* = \{(Hx,Hy) \mid Hx \subset UHx \text{ and } Hy \subset UHy\},
\]

\[
U^- = \{(Hx,Hy) \mid hx \in Uky \text{ for some } h,k \in H\}.
\]

The sets \( U^* \) form a base for the partition uniformity \( \mathcal{U}^* \) on the set \( G/H \) and, if condition \( A \) below is satisfied, then the sets \( U^- \) form a base for a uniformity \( \mathcal{U}^- \).

(A). For each neighborhood \( U \) of \( e \) there is a neighborhood \( V \) such that \( HV \subset UH \).

It is easy to see that \( \mathcal{U}^* \) and \( \mathcal{U}^- \) are right invariant in the sense that they have bases, each element of which is right invariant under action of \( G \). It is known that, if \( H \) is compact, then the topology \( t^* \) induced by \( \mathcal{U}^* \) is equal to the quotient topology [1]. We denote by \( t^- \) the topology induced by the sets \( U^- \). A subset of \( G/H \) is a neighborhood of \( Hx \) in \( t^- \) if it contains a set \( U^-(Hx) = \{Hy \mid (Hx,Hy) \in U^-\} \). If condition \( A \) is satisfied, then \( t^- \) is the topology induced by the uniformity \( \mathcal{U}^- \).

Denote the quotient topology on \( G/H \) by \( q \).

**Lemma.** Each of the following three statements implies the other two:

1. condition \( A \) is satisfied.
2. \( t^- = q \).
3. \( t^* = q \).

**Proof.** We note that \( t^- \subset q \subset t^* \). Suppose condition \( A \) is satisfied and \( U^* \) is any element of the base for \( \mathcal{U}^* \). Let \( V \) be a symmetric neighbor-

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hood of \( e \) such that \( HV \subset UH \). If \((Hx, Hy)\) is in \( V^* \), then \((Hx, Hy) \in U^* \). Hence, \( U^* \supseteq U^* \). Obviously \( U^* \subset U^* \); so (1) implies both (2) and (3).

Now, suppose \( \epsilon = q \) and let \( U \) be any neighborhood of \( e \). There is a neighborhood \( V \) such that \( V^*(H) \subset UH \). But \( V^*(H) = HVH \). Thus, \( HV \subset UH \). It follows that (2) implies (1) and (3). The proof that (3) implies (1) and (2) is as routine as the above.

**Theorem.** If \( G \) is a topological group and \( H \) is a subgroup for which condition \( A \) is satisfied, then \( U^* \supseteq U^* \) is a right invariant uniformity for the quotient space \( G/H \). If \( G/H \) satisfies the first axiom of countability, this uniformity is pseudo-metrizable (metrizable if \( G \) is Hausdorff and \( H \) is closed). Since the uniformity is right invariant, the pseudo-metric (metric) can be chosen so that it is right invariant. This metric is unique in the sense that each right invariant metric on \( G/H \) has \( U^* \) as its uniformity.

**Proof.** By virtue of the above lemma and remarks it is sufficient to point out that a right invariant uniformity for \( G/H \) has a countable base if the topology it induces satisfies the first axiom of countability.

It is obvious that condition \( A \) is satisfied when \( H \) is compact. Thus, the theorem above includes Wu’s and Hudson’s theorems. Actually, compactness implies a stronger condition.

**Remark.** If \( C \) is a compact subset of a topological group \( G \), then, for each neighborhood \( U \) of \( e \), there is a neighborhood \( V \) of \( e \) such that \( xV \subset Ux \) for all \( x \in C \).

This follows from a straightforward argument using nets.

To see other situations in which the theorem holds we note that, if the left uniformity of \( G \) is equal to the right uniformity or the cosets \( Hx, x \in G \), form a star-closed partition of \( G \), then condition \( A \) is satisfied. If the partition is star-closed and \( U \) is an open neighborhood of \( e \), then there is a neighborhood \( V \) of \( e \) such that \( HV \), the saturation of \( V \), is contained in \( UH \); so condition \( A \) is satisfied.

**REFERENCES**