ON A PROBLEM OF ALFSEN AND FENSTAD

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In [2] the authors closed with the following question: Does every p-equivalence class of uniform structures have a finest member? The purpose of the present note is to give a negative answer to this question. Thus completion of proximity spaces is not equivalent to completion of uniform spaces.

Let \((X,p)\) be a general proximity space [3]. Let \(\mathcal{U}\) be the class of all pseudometrics \(\varrho\) on \(X \times X\) which satisfy

\[
\varrho(A,B) = 0 \quad \text{for all subsets } A,B \text{ of } X \text{ with } A \sim p B.
\]

(\(\mathcal{U}\) is the "gauge system" of [5].) Let \(\mathcal{F}\) consist of all totally bounded pseudometrics in \(\mathcal{U}\). We shall consider uniform structures to be classes of pseudometrics with the appropriate properties (see [4, Chapter 15]). From this point of view \(\mathcal{F}\) is a uniform structure [1]. We shall prove (Theorem 2) that \(\mathcal{U}\) need not be a uniform structure.

From [1] it follows that a uniform structure \(\mathcal{S}\) belongs to the equivalence class determined by \(p\) if, and only if,

\[
\mathcal{F} \subseteq \mathcal{S} \subseteq \mathcal{U}.
\]

**Lemma I.** Let \(\mathcal{R}\) be any non-empty subclass of \(\mathcal{U}\) such that

\[
\varrho_1 \text{ and } \varrho_2 \text{ in } \mathcal{R} \implies \varrho_1 \vee \varrho_2 \text{ is in } \mathcal{R}.
\]

Then the uniform structure \(\mathcal{S}\) generated by \(\mathcal{R}\) is a subclass of \(\mathcal{U}\).

**Proof.** In view of (3), \(\mathcal{S}\) consists of all pseudometrics which are uniformly continuous with respect to \(\mathcal{R}\). Since \(\mathcal{U}\) contains every pseudometric uniformly continuous with respect to \(\mathcal{U}\) and since \(\mathcal{R}\) is contained in \(\mathcal{U}\), \(\mathcal{U}\) contains every pseudometric uniformly continuous with respect to \(\mathcal{R}\).

**Lemma II.** Given any pseudometric \(\varrho\) in \(\mathcal{U}\) there exists a uniform structure \(\mathcal{S}\) containing \(\varrho\) such that (2) holds.

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Proof. Let $R$ consist of all $q \vee \beta$ with $\beta$ in $T$. By Lemma 1 of [5], $R$ is contained in $U$. Thus Lemma II follows from Lemma I.

Theorem 1. For a general proximity space the following conditions are equivalent:

(i) The equivalence class of uniform structures determined by $p$ has a finest (i.e. largest) member.

(ii) $U$ is a uniform structure.

(iii) $q_1$ and $q_2$ in $U$ imply $q_1 \vee q_2$ is in $U$.

Proof. The equivalence of (i) and (ii) follows from (2) and Lemma II. That (iii) implies (ii) follows from Lemma I. The converse is a consequence of the definition [4] of uniform structure.

Theorem 2. There exist proximity spaces for which the conditions (i), (ii), (iii) fail to hold.

Proof. Let $X$ be the cartesian product $X_1 \times X_2$ where $X_1 = X_2$ is any infinite set. Let $P_t$ be the canonical projection of $X$ onto $X_t$:

$$P_t x = x_t \quad \text{for} \quad x = (x_1, x_2).$$

For $A, B$ subsets of $X$ define $A \ p \ B$ to mean:

Given any finite coverings $A_1, \ldots, A_m$ of $A$

and $B_1, \ldots, B_n$ of $B$ there exist $A_t$ and $B_j$

such that $P_t A_t$ meets $P_j B_j$ for $t = 1, 2$.

One can verify directly that $p$ is a proximity relation. (Indeed $p$ is the product proximity relation over the product of two discrete proximity spaces [6], [1].) Now $p$ is not the discrete proximity relation. In particular, for $D$ the diagonal in $X$ we contend

$$D \ p \ X - D.$$

To prove (6) consider (5) with $A = D$ and $B = X - D$. Since $D$ is infinite, some $A_t$ from the given covering of $D$ must contain at least two distinct points $(x_1, x_1)$ and $(x_2, x_2)$ of $D$. Thus $(x_1, x_2)$ is in $X - D$, hence in some $B_j$ from the given covering of $X - D$. Thus for $t = 1, 2$ we have $x_t$ in both $P_t A_t$ and $P_j B_j$. So (5) holds, giving (6).

Now we contend that (iii) of Theorem 1 fails to hold for the class $U$ of pseudometrics defined by (1). To show this define for $x = (x_1, x_2)$ and $y = (y_1, y_2)$

$$q_t(x, y) = \begin{cases} 0 & \text{if } x_t = y_t \\ 1 & \text{if } x_t \neq y_t. \end{cases}$$
Clearly each \( q_t \) is in \( \mathcal{U} \) since by (5) \( A \ p \ B \) implies \( P_t A \) meets \( P_t B \), which by (7) implies \( q_t(A, B) = 0 \). Now for \( q = q_1 \vee q_2 \) we have

\[
q(x, y) = \begin{cases} 
0 & \text{if } x = y \\
1 & \text{if } x \neq y .
\end{cases}
\]

Thus,

\[
q(D, X - D) = 1 .
\]

Comparison of (9) with (6) shows that \( q \) is not in \( \mathcal{U} \) since (1) fails to hold.

**Note added in proof:** Theorem 2 has been proved by Alfsen and Njåstad in *Proximity and generalized uniformity*, Fund. Math. 52 (1963), 235–252.

**References**


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