REMOVABLE SINGULARITIES
OF CONTINUOUS HARMONIC FUNCTIONS IN $\mathbb{R}^m$

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1.

Let $E$ be a compact set of $m$-dimensional Euclidean space $\mathbb{R}^m$. If $h(x)$ is harmonic and bounded in a neighbourhood of $E$, and if the $(m-2)$-capacity of $E$ vanishes then $u(x)$ can be extended as a harmonic function to $E$. We say that $E$ is removable for the class of bounded harmonic functions. On the other hand, if $E$ is a smooth closed surface, that is a $(m-1)$-dimensional set, there exist harmonic functions with arbitrarily high smoothness, which cannot be extended to $E$. Our aim here is to prove a theorem which connects the above mentioned two results.

Let $D$ be a domain bounded by a smooth outer surface $\Gamma$ and a compact set $E$ and denote by $H_\alpha$ the class of harmonic functions in $D$ which satisfy a Hölder condition of order $\alpha$, $0 < \alpha < 1$, in $D$:

$$|u(x) - u(x')| \leq \text{Const.} |x - x'|^\alpha, \quad x, x' \in D.$$  

(1.1)

The set $E$ is said to have $\beta$-dimensional measure zero, $0 < \beta < m$, if $E$ can be covered by open spheres of radii $r_\epsilon$ such that $\sum r_\epsilon^\beta$ is arbitrarily small. The following theorem holds.

Theorem. $E$ is removable for the class $H_\alpha$ if and only if $E$ has $(m-2+\alpha)$-dimensional measure zero.

2.

We first assume the $(m-2+\alpha)$-dimensional measure does not vanish. It is then well-known (see Frostman [1]) that there is a distribution $\mu$ of unit mass on $E$ such that

$$\mu(S) \leq C r^{m-2+\alpha}$$

for all spheres $S$ where $r$ denotes the radius of $S$. We shall prove that

$$u(x) = \int_E \frac{d\mu(y)}{|x - y|^{m-2}}$$

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satisfies (1.1). We define \( \mu(r, x) = \mu(\{y \mid |y - x| < r\}) \) and find for \( x, x' \in D, |x - x'| = \delta, \)

\[
u(x) - u(x') = \int_0^\infty r^{2-m} d\mu(r, x) - \int_0^\infty r^{2-m} d\mu(r, x')
\]

\[
= (m - 2) \int_0^\infty (\mu(r, x) - \mu(r, x')) r^{1-m} dr
\]

\[
\leq C_1 \int_0^{2\delta} r^{m-2+\alpha} r^{1-m} dr + (m - 2) \int_0^\infty (\mu(r, x) - \mu(r - \delta, x)) r^{1-m} dr
\]

\[
< C_2 \delta^\alpha + (m - 2) \int_0^\infty (\mu(r, x) r^{1-m} - (r + \delta)^{1-m}) dr
\]

\[
< C_2 \delta^\alpha + C_3 \int_0^{\frac{m-2+\alpha}{1-m}} \frac{\delta}{r^m} dr = C_4 \delta^\alpha.
\]

Since \( x \) and \( x' \) can be interchanged we have proved (1.1).

3.

We now assume that the \((m - 2 + \alpha)\)-dimensional measure of \( E \) vanishes and that \( u(x) \) satisfies (1.1). Let \( u_1(x) \) be the harmonic function inside \( \Gamma \) which is equal to \( u(x) \) on \( \Gamma \). If we define \( v(x) = u(x) - u_1(x) \) then \( v(x) = 0 \) on \( \Gamma \) and our assertion is that \( v(x) \equiv 0 \).

We can cover \( E \) by a finite number of closed spheres \( S_r \),

\[
S_r : \quad |x - x_r| \leq r_v
\]

such that

\[
\sum r_v^{m-2+\alpha} \leq \varepsilon.
\]

We assume that \( \varepsilon \) has its smallest value when the number of spheres is \( \leq n \). In the proof we shall also use the expanded spheres

\[
S_{r_v}(t) : \quad |x - x_r| \leq r_v t, \quad 1 \leq t \leq 3.
\]

For \( t > 1 \) every point of \( E \) is strictly inside \( \bigcup S_{r_v}(t) = \Sigma(t) \). The part of \( \partial \Sigma(t) \) which is boundary of the unbounded component of the complement of \( \Sigma(t) \) is denoted \( \sigma(t) = \bigcup \sigma_v(t) \), where \( \sigma_v(t) \) is \( \sigma(t) \cap \partial S_{r_v}(t) \). Clearly \( \sigma(t) \) does not meet \( E \).

By Green's formula we have, \( t > 1 \),
(3.1) \[ \psi(t) = \int_{D-\Sigma(t)} |\nabla v|^2 \, dx = \int_{\sigma(t)} v \frac{\partial v}{\partial n} \, d\sigma = \frac{1}{2} \int_{\sigma(t)} \frac{\partial v^2}{\partial n} \, d\sigma. \]

If \( v \equiv \text{constant}, \) \( \psi(t) \) is \( \geq \text{const.} > 0 \) in \( 1 < t \leq 3, \) if \( \varepsilon \) is small enough. We rewrite (3.1) introducing the unit sphere \( U. \) Points on \( U \) are denoted \( \xi \) and its area element \( dA_\xi. \) The part of \( U \) for which \( x_\nu + tr_\nu \xi \in \sigma_\nu(t) \) is called \( a_\nu(t). \) Integrating (3.1) and using these notations we find

(3.2) \[ -2 \sum_{n} r_\nu^{m-2} \int_{a_\nu(t)} \int_0^t \frac{\partial}{\partial t} v^2(x_\nu + tr_\nu \xi) \, dA_\xi. \]

In each term on the right of (3.2) we shall now interchange the order of integration. We must then study for \( \xi \) fixed for which values of \( t \) a certain ray \( x_\nu + tr_\nu \xi \) belongs to \( \sigma_\nu(t). \) We distinguish four cases, the first two of which are trivial.

(a) \( x_\nu + tr_\nu \xi \in \sigma_\nu(t), \ 2 \leq t \leq 3. \) For such a \( \xi \) we get 0 as contribution to (3.2) from the \( \nu \)th term.

(b) \( x_\nu + tr_\nu \xi \in \sigma_\nu(t), \ 2 \leq t \leq 3. \) We can evaluate the \( t \)-integration and get the contribution \( v^2(x_\nu + 3r_\nu \xi) - v^2(x_\nu + 2r_\nu \xi) = O(r_\nu^\alpha). \)

(c) The remaining possibility is: \( x_\nu + tr_\nu \xi \in \sigma_\nu(t), \ \tau_\nu \leq t \leq \tau_\nu', \ i = 0, 1, 2, \ldots, p, \ 2 \leq \tau_0 < \tau_0' < \tau_1 < \ldots < \tau_p' \leq 3. \) For every \( \tau_\nu', \ i < p, \) there is an index \( \mu + \nu \) so that \( x_\nu + tr_\nu \tau_\nu \xi \in \sigma_\nu(t_\nu'). \) We here have two essentially different cases.

(c1) \( r_\mu \geq r_\nu. \) If we consider the two-dimensional plane containing \( x_\nu, x_\mu \) and \( x = x_\nu + r_\nu \tau_\nu \xi, \) we see that \( x' = x_\nu + tr_\nu \xi, \ t > \tau_\nu', \) must be interior to \( S_\mu(t) \) and hence \( x_\nu + tr_\nu \xi \in \sigma_\nu(t), \ t > \tau_\nu'. \) (c1) can thus occur only if \( i = p. \)

(c2) We now assume \( i < p \) and \( r_\mu \geq r_\nu. \) We first observe that \( x_\nu + tr_\nu \xi, \ 2 \leq t \leq 3, \) belongs to a certain sphere \( S_\mu(t) \) in a \( t \)-interval and that its length is \( \leq 6r_\mu r_\nu^{-1}. \) We now consider an interval \( (\tau_\nu', \tau_{\nu+1}). \) The corresponding spheres are \( S_\mu(t), \ \mu = \mu_1, \ldots, \mu_k. \) We can write, if \( \varphi(t) = v^2(x_\nu + tr_\nu \xi), \)

(3.3) \[ \varphi(\tau_{\nu+1}) - \varphi(\tau_{\nu}) = \sum_{j=1}^{k} (\varphi(s_{j+1}) - \varphi(s_j)), \]

where each pair \( s_j, s_{j+1} \) belongs to one \( S_\mu(3). \) Hence

(3.4) \[ |\varphi(\tau_{\nu+1}) - \varphi(\tau_{\nu})| \leq \sum_{j=1}^{k} C r_\nu^\alpha |s_{j+1} - s_j|^\alpha \leq C 6^\alpha \sum_{j=1}^{k} r_\mu^\alpha, \]

We now evaluate the \( t \)-integral of the \( \nu \)th term in (3.2) and find

\[ \sum_{\nu=0}^{p} (\varphi(\tau_{\nu}) - \varphi(\tau_{\nu})). \]
If we add the relations (3.3) for $i = 0, 1, \ldots, p-1$, and use (3.4) we get the estimate
\begin{equation}
O(r_v^\alpha) + O(\Sigma^1 r_\mu^\alpha),
\end{equation}
where $\Sigma^1$ indicates that the summation is extended over those $\mu$ for which $x_\mu + r_\mu t \xi$, $t \leq 3$, meets $S_\mu(3)$.

We consider the estimate (3.5) for different points $\xi \in U$. The area of $U$ for which $x_\mu + r_\mu t \xi \in S_\mu(3)$ for some $t$ is $O(r_\mu^{m-1} r_\nu^{1-m})$. The total $\nu$th term in (3.2) is thus
\begin{equation}
O(r_v^{m-2+\alpha}) + O(r_v^{-1} \Sigma^2 r_\mu^{m-1+\alpha}),
\end{equation}
where $\Sigma^2$ indicates summation over those $\mu$ for which $S_\mu(3) \cap S_\nu(3) \neq \emptyset$ and $r_\mu \leq r_\nu$. The last relations imply $S_\mu(1) \subset S_\nu(7)$. Since the covering by the spheres $S_\nu = S_\nu(1)$ was assumed to be minimal we have
\begin{equation*}
\Sigma^2 r_\mu^{m-2+\alpha} \leq 7^{m-2+\alpha} r_v^{m-2+\alpha}.
\end{equation*}

If we use this and $r_\mu \leq r_v$ in (3.6), we find that the $\nu$th term in (3.2) is $O(r_v^{m-2+\alpha})$ and so
\begin{equation*}
\int_2^3 \psi(t) t^{1-m} dt \leq \text{Const} \sum_1^n r_v^{m-2+\alpha} \leq \text{Const} \cdot \varepsilon.
\end{equation*}

Hence $\psi(t)$ cannot be uniformly positive and so $\nu(x) \equiv \text{constant}$, and then $\nu \equiv 0$, as was to be proved.

REFERENCE


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