MATH. SCAND. 12 (1963), 5–11

A THEOREM ON THE MINIMUM MODULUS OF ENTIRE FUNCTIONS

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1. Introduction.

Let $f(z)$ be an entire function. We denote $\max|f(z)|$ and $\min|f(z)|$ on $|z|=r$ by $M(r)$ and $m(r)$, respectively. The order and lower order are defined as lim sup and lim inf of $\log\log M(r)/\log r$ as $r \to \infty$. Many papers have been devoted to the problem of finding relations between $M(r)$ and $m(r)$. Surveys of this subject have been given by Hayman [3] and also by Goldberg and Ostrovskij [2]. As a refinement of previous results by the author [6] we intend to prove the following theorem.

THEOREM. *For each non-constant entire function $f(z)$ and for each number λ satisfying $0<\lambda<1$ the following holds: Either*

$$\log m(r) > \cos\pi\lambda \log M(r) \tag{1}$$

for certain arbitrarily large values of r, or, if (1) is not fulfilled, the limit

$$\lim_{r\to\infty} \frac{\log M(r)}{r^{\lambda}} \tag{2}$$

exists and is positive or is infinite.

In the case of $\lambda=\frac{1}{2}$ the theorem has been proved by Heins [5]. In the review of the paper [6] by the present author, Hayman [4] made a conjecture ($\alpha=\beta$ below) which together with the results in [6] constitutes the above theorem.

2. Preliminary discussion.

To prove the theorem we shall suppose that (1) does not hold. Then we have

$$\log m(r) \leqq \cos\pi\lambda \log M(r) \tag{3}$$

perhaps not for all $r \geqq 0$ but in any case for all $r \geqq r_0 \geqq 0$. When $m(r)=0$ we still consider $\log m(r)$ as defined and having the value $-\infty$.

In [6] we have proved that the inequality (3) implies that (see [6, Theorem II])

Received January 8, 1963.

$$\alpha = \liminf_{r\to\infty} \frac{\log M(r)}{r^\lambda} > 0 . \tag{4}$$

Here ">0" means that the limit is a positive number or $+\infty$.

Consider first the case when the limit is a positive number, i.e.

$$0 < \alpha < \infty . \tag{5}$$

As has been observed in the last section of [6], the relations (3) and (5) also imply that

$$\beta = \limsup_{r\to\infty} \frac{\log M(r)}{r^\lambda} < \infty . \tag{6}$$

In the remaining case we have

$$\liminf_{r\to\infty} \frac{\log M(r)}{r^\lambda} = +\infty .$$

Then

$$\lim_{r\to\infty} \frac{\log M(r)}{r^\lambda} = +\infty \tag{7}$$

and the alternative (2) of the theorem is fulfilled. Thus the theorem stated at the beginning will be proved if we can show that (3), (5) and (6) imply that $\alpha=\beta$.

To be exact, the paper [6] mentioned above contains nothing about entire functions of order zero. In this well-known case, however, it is true that $\log m(r) \sim \log M(r)$ for certain arbitrarily large values of r (cf. Boas [1, Theorem 3.6.2]). Thus, for these functions the alternative (1) is always fulfilled.

It would be convenient to have the inequality (3) satisfied not only for $r \geqq r_0$ but for $r \geqq 0$. This can be achieved by dividing the function in question by a constant $C>1$. To determine C we first observe that $\log M(r)$ is continuous for $r>0$ and that $\log m(r)$ is upper semi-continuous for $r \geqq 0$, i.e. for each real number q the set where $\log m(r) < q$ is an open set. Thus

$$\varphi(r) = \log m(r) - \cos\pi\lambda \log M(r)$$

is upper semi-continuous for $r>0$. Further, as $r \to 0$,

$$\varphi(r) \to (1-\cos\pi\lambda) \log|f(0)| .$$

This means that in some interval $0 \leqq r < \delta$ the function $\varphi(r)$ is continuous if $f(0) \neq 0$, upper semi-continuous if $f(0)=0$. The result is that $\varphi(r)$ is upper semi-continuous for $r \geqq 0$. Suppose that a function $f(z)$ does not satisfy (3) for $r \geqq 0$ but only for $r \geqq r_0 > 0$. Then the upper semi-continuity of $\varphi(r)$ implies that $\varphi(r)$ attains a positive maximum for $0 \leqq r \leqq r_0$. Denote this maximum by $(1-\cos\pi\lambda)\log C$. Then

$$\varphi(r)-(1-\cos\pi\lambda)\log C = \log\frac{m(r)}{C} - \cos\pi\lambda\log\frac{M(r)}{C} \leqq 0 \tag{8}$$

holds for all $r \geqq 0$.

Later on it will also be a little more convenient for the proof if the modulus of the entire function at the origin is less than one. If necessary, we therefore take a somewhat larger value of C than is needed to fulfil (8). Of course, dividing by a constant does not affect the values of α and β in (5) and (6).

Summing up, we are going to consider an entire function $f(z)$ satisfying (5) and (6). These relations imply that $f(z)$ is of order λ, where $0<\lambda<1$, and of "very regular growth". Every function of order less than one can be represented as an infinite product in the following way:

$$f(z) \equiv Az^p \prod_1^\infty (1-z/a_n)\,, \tag{9}$$

where $A \neq 0$ is a constant, p is a non-negative integer and $a_1, a_2, \ldots, a_n, \ldots$ are the zeros outside the origin. We assume that the division, if necessary, by a constant is already done so that (3) holds for all $r \geqq 0$ and also that

$$|f(0)| < 1\,. \tag{10}$$

We then want to prove that $\alpha=\beta$.

3. An integral inequality.

To begin with we perform some calculations valid for each entire function $f(z)$ of order ϱ less than one. Such a function can be represented as in (9). As usual we also form an auxiliary function $f_1(z)$ with real and non-positive zeros:

$$f_1(z) \equiv |A| z^p \prod_1^\infty (1+z/|a_n|)\,. \tag{11}$$

The maximum and minimum of $|f_1(z)|$ on $|z|=r$ are denoted by $M_1(r)$ and $m_1(r)$, just as $M(r)$ and $m(r)$ for $f(z)$. Then for all $r \geqq 0$ we know that (cf. Boas [1, 3.2])

$$m_1(r) = |f_1(-r)| \leqq m(r) \leqq M(r) \leqq f_1(r) = M_1(r) \tag{12}$$

holds and also

$$m_1(r)M_1(r) \leqq m(r)M(r)\,. \tag{13}$$

The function $\log|f_1(z)|$ is harmonic in the plane cut along the negative real axis. Like $f(z)$, the function $f_1(z)$ is also of order ϱ (cf. Boas [1, Theorem 2.9.5]). The magnitude of $\log|f_1(z)|$ is then small enough to permit a representation of $\log|f_1(z)|$ by means of its boundary values in each half-plane where it is harmonic (cf. Boas [1, 6.5]). Considering the upper half-plane we thus get at a point iy, $y>0$:

$$\log|f_1(iy)| = \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{\log|f_1(x)|}{x^2+y^2}\,dx$$

or

$$\log|f_1(iy)| = \frac{y}{\pi}\int_0^{\infty}\frac{\log m_1(r)+\log M_1(r)}{r^2+y^2}\,dr\,. \tag{14}$$

Because of the symmetry of $f_1(z)$ we have $\log|f_1(-iy)|=\log|f_1(iy)|$. If we then apply the representation formula in the right half-plane we obtain for $R>0$

$$\begin{aligned}\log M_1(R) &= \frac{2R}{\pi}\int_0^{\infty}\frac{\log|f_1(iy)|}{y^2+R^2}\,dy \\ &= \frac{2R}{\pi}\int_0^{\infty}\frac{dy}{y^2+R^2}\,\frac{y}{\pi}\int_0^{\infty}\frac{\log m_1(r)+\log M_1(r)}{r^2+y^2}\,dr \\ &= \frac{2R}{\pi^2}\int_0^{\infty}\{\log m_1(r)+\log M_1(r)\}\,dr\int_0^{\infty}\frac{y\,dy}{(y^2+R^2)(y^2+r^2)} \\ &= \frac{2R}{\pi^2}\int_0^{\infty}\frac{\log m_1(r)+\log M_1(r)}{R^2-r^2}\log\frac{R}{r}\,dr\,.\end{aligned} \tag{15}$$

By (12) we have

$$\log M_1(R) \geqq \log M(R)$$

and by (13)

$$\log m_1(r)+\log M_1(r) \leqq \log m(r)+\log M(r)$$

If we apply these last inequalities and also divide by R^{λ} we obtain from (15)

$$\frac{\log M(R)}{R^{\lambda}} \leqq \frac{2}{\pi^2}\int_0^{\infty}\frac{\log m(r)+\log M(r)}{r^{\lambda}}\left(\frac{r}{R}\right)^{\lambda}\frac{R\log(R/r)}{R^2-r^2}\,dr, \tag{16}$$

valid for entire functions of order less than one.

Let us also suppose that (3) holds for all $r\geqq 0$. Because the kernel function $K(r,R)$ defined below is non-negative we then obtain from (16) the following formula, fundamental for our proof:

$$\frac{\log M(R)}{R^\lambda} < \int_0^\infty \frac{\log M(r)}{r^\lambda} K(r,R)\, dr. \tag{17}$$

The expression for the kernel function $K(r,R)$ is

$$K(r,R) = \frac{2(1+\cos\pi\lambda)}{\pi^2} \left(\frac{r}{R}\right)^\lambda \frac{R\log(R/r)}{R^2-r^2}. \tag{18}$$

A residue calculation shows that

$$\int_0^\infty K(r,R)\, dr = 1. \tag{19}$$

We have strict inequality in (17) because we have strict inequality in (3) in certain intervals, for instance in the neighbourhood of the zeros of $f(z)$.

4. The final proof.

Let us consider an entire function $f(z)$ satisfying (3) for all $r \geqq 0$, (5), (6), (9) and (10). We then have the integral inequality (17), which provides the key to the solution of our problem (to show that $\alpha = \beta$).

Because of (10) the function

$$\psi(r) = \frac{\log M(r)}{r^\lambda} \to -\infty \tag{20}$$

as $r \to 0$. We may define its value as $-\infty$ for $r=0$. Then $\psi(r)$ is continuous for $r>0$ and upper semi-continuous for $r \geqq 0$. Thus it attains a maximum in each closed sub-interval of $r \geqq 0$. But $\psi(r)$ does not attain any largest value in the infinite interval $r \geqq 0$. To understand this, suppose a largest value ψ_0 be attained at R_0. Then (17) and (19) would give the impossible relation $\psi_0 < \psi_0$. Consequently, it must be true that

$$\psi(r) < \beta \tag{21}$$

for all $r \geqq 0$.

Let R_1 be a value of r so large that $\psi(r)$ has a positive maximum b in the interval $(0, R_1)$. Let R be a value of r in this interval where the maximum is attained, i.e.

$$b = \max_{0 \leqq r \leqq R_1} \psi(r) = \psi(R) > 0, \qquad b < \beta. \tag{22}$$

We also set

$$(23) \qquad a = \psi(R_1) > 0 .$$

Because $\log M(r)$ is an increasing function, $b>0$ yields $a>0$. Let us define

$$(24) \qquad k = (a/b)^{1/(2\lambda)} .$$

In the interval $kR_1 \leqq r \leqq R_1$ we have

$$(25) \qquad \psi(r) = \frac{\log M(r)}{r^\lambda} \leqq \left(\frac{R_1}{r}\right)^\lambda \frac{\log M(R_1)}{R_1{}^\lambda} \leqq \frac{a}{k^\lambda} = (ab)^{\frac{1}{2}} .$$

Let us now return to the integral inequality (17). In the right-hand side of (17) we use the following upper bounds of $\psi(r)$ according to (22), (25) and (21):

For $0 \leqq r \leqq kR_1$ we have $\psi(r) \leqq b$.
For $kR_1 \leqq r \leqq R_1$ we have $\psi(r) \leqq (ab)^{\frac{1}{2}} = b-(b-(ab)^{\frac{1}{2}})$.
For $R_1 \leqq r$ we have $\psi(r) < \beta = b+(\beta-b)$.

By choosing R as in (22) we then get from (17) that

$$(26) \quad b < b\int_0^\infty K(r,R)\,dr - \left(b-(ab)^{\frac{1}{2}}\right)\int_{kR_1}^{R_1} K(r,R)\,dr + (\beta-b)\int_{R_1}^\infty K(r,R)\,dr .$$

Because of (19) we obtain

$$(27) \qquad \left(b-(ab)^{\frac{1}{2}}\right)\int_{kR_1}^{R_1} K(r,R)\,dr < (\beta-b)\int_{R_1}^\infty K(r,R)\,dr .$$

We can choose arbitrarily large values of R_1 such that

$$a \approx \alpha, \qquad b \approx \beta .$$

Let us now suppose that $\alpha<\beta$. A rough estimation of the integrals in (27) gives

$$(28) \qquad \int_{kR_1}^{R_1} K(r,R)\,dr = \frac{2(1+\cos\pi\lambda)}{\pi^2}\int_{kc}^{c}\frac{t^\lambda \log t}{t^2-1}\,dt$$
$$> (1+\cos\pi\lambda)(1-k)\,\frac{\log c}{c^{1-\lambda}}$$

and

$$(29) \qquad \int_{R_1}^\infty K(r,R)\,dr < \frac{2(1+\cos\pi\lambda)}{\pi^2(1-k^2)}\left\{\frac{1}{1-\lambda}-\frac{1}{(1-\lambda)^2\log k}\right\}\cdot\frac{\log c}{c^{1-\lambda}}$$

where

$$c = R_1/R > 1/k .$$

It is now obvious that (27) contradicts (28) and (29) since we supposed that $\alpha < \beta$. Therefore $\alpha = \beta$, and the theorem is proved.

After having read this paper in manuscript L. Carleson made a remark concerning the conclusion in Section 4 above from the integral inequality (17). The result can be obtained from a general theory of integral inequalities by Matts Essén. We find this very interesting and it has been arranged so that a separate proof by Essén immediately follows in this journal.

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