A THEOREM ON THE MINIMUM MODULUS OF
ENTIRE FUNCTIONS

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1. Introduction.

Let \( f(z) \) be an entire function. We denote \( \max |f(z)| \) and \( \min |f(z)| \)
on \( |z| = r \) by \( M(r) \) and \( m(r) \), respectively. The order and lower order
are defined as \( \limsup \) and \( \liminf \) of \( \log \log M(r) / \log r \) as \( r \to \infty \). Many
papers have been devoted to the problem of finding relations between \( M(r) \) and \( m(r) \). Surveys of this subject have been given by Hayman [3]
and also by Goldberg and Ostrovskij [2]. As a refinement of previous
results by the author [6] we intend to prove the following theorem.

**Theorem.** For each non-constant entire function \( f(z) \) and for each number
\( \lambda \) satisfying \( 0 < \lambda < 1 \) the following holds: Either

\[
(1) \quad \log m(r) > \cos \pi \lambda \log M(r)
\]

for certain arbitrarily large values of \( r \), or, if (1) is not fulfilled, the limit

\[
(2) \quad \lim_{r \to \infty} \frac{\log M(r)}{r^\lambda}
\]

exists and is positive or is infinite.

In the case of \( \lambda = \frac{1}{2} \) the theorem has been proved by Heins [5]. In
conjecture \( (\alpha = \beta \) below) which together with the results in [6] constitu-
tes the above theorem.

2. Preliminary discussion.

To prove the theorem we shall suppose that (1) does not hold. Then
we have

\[
(3) \quad \log m(r) \leq \cos \pi \lambda \log M(r)
\]

perhaps not for all \( r \geq 0 \) but in any case for all \( r \geq r_0 \geq 0 \). When \( m(r) = 0 \)
we still consider \( \log m(r) \) as defined and having the value \( -\infty \).

In [6] we have proved that the inequality (3) implies that (see [6, Theorem II])

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\( \alpha = \liminf_{r \to \infty} \frac{\log M(r)}{r^\lambda} > 0. \)

Here "\( > 0 \)" means that the limit is a positive number or \( +\infty \).

Consider first the case when the limit is a positive number, i.e.

\( 0 < \alpha < \infty. \)

As has been observed in the last section of [6], the relations (3) and (5) also imply that

\( \beta = \limsup_{r \to \infty} \frac{\log M(r)}{r^\lambda} < \infty. \)

In the remaining case we have

\( \liminf_{r \to \infty} \frac{\log M(r)}{r^\lambda} = +\infty. \)

Then

\( \lim_{r \to \infty} \frac{\log M(r)}{r^\lambda} = +\infty \)

and the alternative (2) of the theorem is fulfilled. Thus the theorem stated at the beginning will be proved if we can show that (3), (5) and (6) imply that \( \alpha = \beta. \)

To be exact, the paper [6] mentioned above contains nothing about entire functions of order zero. In this well-known case, however, it is true that \( \log m(r) \sim \log M(r) \) for certain arbitrarily large values of \( r \) (cf. Boas [1, Theorem 3.6.2]). Thus, for these functions the alternative (1) is always fulfilled.

It would be convenient to have the inequality (3) satisfied not only for \( r \geq r_0 \) but for \( r \geq 0 \). This can be achieved by dividing the function in question by a constant \( C > 1 \). To determine \( C \) we first observe that \( \log M(r) \) is continuous for \( r > 0 \) and that \( \log m(r) \) is upper semi-continuous for \( r \geq 0 \), i.e. for each real number \( q \) the set where \( \log m(r) < q \) is an open set. Thus

\( \varphi(r) = \log m(r) - \cos \pi \lambda \log M(r) \)

is upper semi-continuous for \( r > 0 \). Further, as \( r \to 0 \),

\( \varphi(r) \to (1 - \cos \pi \lambda) \log |f(0)|. \)

This means that in some interval \( 0 \leq r < \delta \) the function \( \varphi(r) \) is continuous if \( f(0) \neq 0 \), upper semi-continuous if \( f(0) = 0 \). The result is that \( \varphi(r) \) is upper semi-continuous for \( r \geq 0 \). Suppose that a function \( f(z) \) does not satisfy (3) for \( r \geq 0 \) but only for \( r \geq r_0 > 0 \). Then the upper semi-con-
continuity of \( \varphi(r) \) implies that \( \varphi(r) \) attains a positive maximum for \( 0 \leq r \leq r_0 \). Denote this maximum by \( (1 - \cos \pi \lambda) \log C \). Then

\[
(8) \quad \varphi(r) - (1 - \cos \pi \lambda) \log C = \log \frac{m(r)}{C} - \cos \pi \lambda \log \frac{M(r)}{C} \leq 0
\]

holds for all \( r \geq 0 \).

Later on it will also be a little more convenient for the proof if the modulus of the entire function at the origin is less than one. If necessary, we therefore take a somewhat larger value of \( C \) than is needed to fulfil (8). Of course, dividing by a constant does not affect the values of \( \alpha \) and \( \beta \) in (5) and (6).

Summing up, we are going to consider an entire function \( f(z) \) satisfying (5) and (6). These relations imply that \( f(z) \) is of order \( \lambda \), where \( 0 < \lambda < 1 \), and of "very regular growth". Every function of order less than one can be represented as an infinite product in the following way:

\[
(9) \quad f(z) \equiv Az^p \prod_{1}^{\infty} (1 - z/\alpha_{n}) ,
\]

where \( A \neq 0 \) is a constant, \( p \) is a non-negative integer and \( \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots \) are the zeros outside the origin. We assume that the division, if necessary, by a constant is already done so that (3) holds for all \( r \geq 0 \) and also that

\[
(10) \quad |f(0)| < 1 .
\]

We then want to prove that \( \alpha = \beta \).

3. An integral inequality.

To begin with we perform some calculations valid for each entire function \( f(z) \) of order \( \varphi \) less than one. Such a function can be represented as in (9). As usual we also form an auxiliary function \( f_1(z) \) with real and non-positive zeros:

\[
(11) \quad f_1(z) \equiv |A| z^p \prod_{1}^{\infty} (1 + z/|\alpha_n|) .
\]

The maximum and minimum of \( |f_1(z)| \) on \( |z| = r \) are denoted by \( M_1(r) \) and \( m_1(r) \), just as \( M(r) \) and \( m(r) \) for \( f(z) \). Then for all \( r \geq 0 \) we know that (cf. Boas [1, 3.2])

\[
(12) \quad m_1(r) = |f_1(-r)| \leq m(r) \leq M(r) \leq f_1(r) = M_1(r)
\]

holds and also

\[
(13) \quad m_1(r) M_1(r) \leq m(r) M(r) .
\]
The function \( \log |f_1(z)| \) is harmonic in the plane cut along the negative real axis. Like \( f(z) \), the function \( f_1(z) \) is also of order \( \sigma \) (cf. Boas [1, Theorem 2.9.5]). The magnitude of \( \log |f_1(z)| \) is then small enough to permit a representation of \( \log |f_1(z)| \) by means of its boundary values in each half-plane where it is harmonic (cf. Boas [1, 6.5]). Considering the upper half-plane we thus get at a point \( iy, y > 0 \):

\[
\log |f_1(iy)| = \frac{y}{\pi} \int_{-\infty}^{+\infty} \log |f_1(x)| \frac{dx}{x^2 + y^2}
\]
or

\[
(14) \quad \log |f_1(iy)| = \frac{y}{\pi} \int_{0}^{\infty} \frac{\log m_1(r) + \log M_1(r)}{r^2 + y^2} \, dr.
\]

Because of the symmetry of \( f_1(z) \) we have \( \log |f_1(-iy)| = \log |f_1(iy)| \). If we then apply the representation formula in the right half-plane we obtain for \( R > 0 \)

\[
(15) \quad \log M_1(R) = \frac{2R}{\pi} \int_{0}^{\infty} \frac{\log |f_1(iy)|}{y^2 + R^2} \, dy
\]

\[
= \frac{2R}{\pi} \int_{0}^{\infty} \frac{dy}{y^2 + R^2} \frac{y}{\pi} \int_{0}^{\infty} \frac{\log m_1(r) + \log M_1(r)}{r^2 + y^2} \, dr
\]

\[
= \frac{2R}{\pi^2} \int_{0}^{\infty} \int_{0}^{\infty} \frac{\log m_1(r) + \log M_1(r)}{(y^2 + R^2)(y^2 + r^2)} \frac{y \, dy}{(y^2 + R^2)(y^2 + r^2)}
\]

\[
= \frac{2R}{\pi^2} \int_{0}^{\infty} \frac{\log m_1(r) + \log M_1(r)}{R^2 - r^2} \log \frac{R}{r} \, dr.
\]

By (12) we have

\[
\log M_1(R) \geq \log M(R)
\]

and by (13)

\[
\log m_1(r) + \log M_1(r) \leq \log m(r) + \log M(r)
\]

If we apply these last inequalities and also divide by \( R^4 \) we obtain from (15)

\[
(16) \quad \frac{\log M(R)}{R^4} \leq \frac{2}{\pi^2} \int_{0}^{\infty} \frac{\log m(r) + \log M(r)}{r^4} \frac{(r)}{R} \frac{R \log (R/r)}{R^2 - r^2} \, dr,
\]

valid for entire functions of order less than one.

Let us also suppose that (3) holds for all \( r \geq 0 \). Because the kernel
function $K(r, R)$ defined below is non-negative we then obtain from (16) the following formula, fundamental for our proof:

$$
(17) \quad \frac{\log M(R)}{R^3} < \int_0^\infty \frac{\log M(r)}{r^3} K(r, R) \, dr .
$$

The expression for the kernel function $K(r, R)$ is

$$
(18) \quad K(r, R) = \frac{2(1 + \cos \pi \lambda)}{\pi^2} \left( \frac{r}{R} \right)^\lambda \frac{R \log (R/r)}{R^2 - r^2} .
$$

A residue calculation shows that

$$
(19) \quad \int_0^\infty K(r, R) \, dr = 1 .
$$

We have strict inequality in (17) because we have strict inequality in (3) in certain intervals, for instance in the neighbourhood of the zeros of $f(z)$.

4. The final proof.

Let us consider an entire function $f(z)$ satisfying (3) for all $r \geq 0$, (5), (6), (9) and (10). We then have the integral inequality (17), which provides the key to the solution of our problem (to show that $\alpha = \beta$).

Because of (10) the function

$$
(20) \quad \psi(r) = \frac{\log M(r)}{r^3} \to -\infty
$$

as $r \to 0$. We may define its value as $-\infty$ for $r = 0$. Then $\psi(r)$ is continuous for $r > 0$ and upper semi-continuous for $r \geq 0$. Thus it attains a maximum in each closed sub-interval of $r \geq 0$. But $\psi(r)$ does not attain any largest value in the infinite interval $r \geq 0$. To understand this, suppose a largest value $\psi_0$ be attained at $R_0$. Then (17) and (19) would give the impossible relation $\psi_0 < \psi_0$. Consequently, it must be true that

$$
(21) \quad \psi(r) < \beta
$$

for all $r \geq 0$.

Let $R_1$ be a value of $r$ so large that $\psi(r)$ has a positive maximum $b$ in the interval $(0, R_1)$. Let $R$ be a value of $r$ in this interval where the maximum is attained, i.e.

$$
(22) \quad b = \max_{0 \leq r \leq R_1} \psi(r) = \psi(R) > 0, \quad b < \beta .
$$
We also set
\[(23)\quad a = \psi(R_1) > 0 .\]

Because \(\log M(r)\) is an increasing function, \(b > 0\) yields \(a > 0\). Let us define
\[(24)\quad k = (a/b)^{1/(23)} .\]

In the interval \(kR_1 \leq r \leq R_1\) we have
\[(25)\quad \psi(r) = \frac{\log M(r)}{r^\lambda} \leq \left(\frac{R_1}{r}\right)^{\lambda} \frac{\log M(R_1)}{R_1^\lambda} \leq \frac{a}{k^\lambda} = (ab)^k .\]

Let us now return to the integral inequality (17). In the right-hand side of (17) we use the following upper bounds of \(\psi(r)\) according to (22), (25) and (21):

- For \(0 \leq r \leq kR_1\) we have \(\psi(r) \leq b\).
- For \(kR_1 \leq r \leq R_1\) we have \(\psi(r) \leq (ab)^k = b - (b - (ab)^k)\).
- For \(R_1 \leq r\) we have \(\psi(r) < \beta = b + (\beta - b)\).

By choosing \(R\) as in (22) we then get from (17) that
\[(26)\quad b < b \int_{0}^{\infty} K(r,R) \, dr - (b - (ab)^k) \int_{kR_1}^{R_1} K(r,R) \, dr + (\beta - b) \int_{R_1}^{\infty} K(r,R) \, dr .\]

Because of (19) we obtain
\[(27)\quad (b - (ab)^k) \int_{kR_1}^{R_1} K(r,R) \, dr < (\beta - b) \int_{R_1}^{\infty} K(r,R) \, dr .\]

We can choose arbitrarily large values of \(R_1\) such that
\[a \approx \alpha, \quad b \approx \beta .\]

Let us now suppose that \(\alpha < \beta\). A rough estimation of the integrals in (27) gives
\[(28)\quad \int_{kR_1}^{R_1} K(r,R) \, dr = \frac{2(1 + \cos \pi \lambda)}{\pi^2} \int_{k\lambda}^{c} \frac{t^2 \log t}{t^2 - 1} \, dt > (1 + \cos \pi \lambda)(1 - k) \frac{\log c}{c^{1 - \lambda}}\]

and
\[(29)\quad \int_{R_1}^{\infty} K(r,R) \, dr < \frac{2(1 + \cos \pi \lambda)}{\pi^2(1 - k^2)} \left[ \frac{1}{1 - \lambda} - \frac{1}{(1 - \lambda)^2 \log k} \right] \frac{\log c}{c^{1 - \lambda}} .\]
where
\[ c = \frac{R_1}{R} > \frac{1}{k}. \]

It is now obvious that (27) contradicts (28) and (29) since we supposed that \( \alpha < \beta \). Therefore \( \alpha = \beta \), and the theorem is proved.

After having read this paper in manuscript L. Carleson made a remark concerning the conclusion in Section 4 above from the integral inequality (17). The result can be obtained from a general theory of integral inequalities by Matts Essén. We find this very interesting and it has been arranged so that a separate proof by Essén immediately follows in this journal.

REFERENCES


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