PERTURBATION OF
ORDINARY DIFFERENTIAL OPERATORS\textsuperscript{1}

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1. Introduction.

In the present paper \( L \) will denote an operator in the Hilbert space \( \mathcal{L}^2(a, b) \) derived from a formally selfadjoint, in general singular, ordinary differential operator of order \( 2n \) and \( B \) will denote an operator of the form \( By = \beta y^{(k)} \) of order \( k \), \( 0 \leq k \leq 2n - 1 \).

We find conditions under which, in the terminology of Wolf [6] the operator \( B \) is \( L \)-compact. The study of such conditions is of interest in view of the fact that perturbation of the operator \( L \) by an \( L \)-compact operator leaves the essential spectrum unchanged (the spectrum of \( L \) is divided into the set of isolated eigenvalues of finite multiplicity and the rest of the spectrum, which is called the essential spectrum). For the basic results concerning \( L \)-compact perturbations we refer to the paper by Gohberg and Krein [2] in which general Banach spaces are considered, and to the paper by Wolf [6], where a simpler and more detailed treatment is given for the case of Hilbert spaces.

As was pointed out to the author by professor Kuroda (see Kuroda [4]), it is of importance for some questions of quantum mechanics to know whether \( B \) is of \( L \)-Hilbert–Schmidt type (see definition 3.5), and in the theorems 5I,1, 5II,2 and 5II,3 we give necessary and sufficient conditions for this.

It turns out that a necessary condition that \( B \) be defined on the domain of \( L \) is that \( \beta \in \mathcal{L}^2_{\text{loc}}(a, b) \), and the sufficient conditions for \( B \) to be \( L \)-compact or of \( L \)-Hilbert–Schmidt type are growth conditions on

\[ \int_{\alpha}^{\beta} |\beta(x)|^2 \, dx \]

for \( \alpha \to a \) and \( \beta \to b \).

The main results are formulated in the theorems of section 5: I 1,3,4; II 2,3,5,6 and III 2,3,5,7.

\textsuperscript{1} This research has been partially supported by a grant of the U. S. National Science Foundation.
I wish to express my thanks to E. Thue Poulsen, who suggested the problem to me, and to Professor F. Wolf for helpful conversations and suggestions during the course of the present work.

2. Definition of the operators.

a) The unperturbed operators. The terminology of Neumark [5, Kap. V] will be used.

Let \((a, b)\) be an interval, where \(a = -\infty\) and \(b = +\infty\) are allowed as boundary points, and let

\[ p_0, p_1, \ldots, p_n \]

be real-valued functions on \((a, b)\), such that \(1/p_0, p_1, \ldots, p_n\) are locally integrable. The quasi-derivatives \(y^{[k]}\) of a complex-valued function \(y\) on \((a, b)\) are defined by

\[
y^{[k]} = \frac{d^k y}{dx^k} \quad \text{for} \quad k = 0, 1, \ldots, n - 1,
\]

\[
y^{[n]} = p_0 \frac{d^n y}{dx^n},
\]

\[
y^{[n+k]} = p_k \frac{d^{n-k} y}{dx^{n-k}} - \frac{d}{dx} y^{[n+k-1]} \quad \text{for} \quad k = 1, 2, \ldots, n.
\]

The formal differential operator \(l\) is defined by

\[
D(l) = \{ y \mid y^{[k]} \text{exist, loc. a.c., for } 0 \leq k \leq 2n - 1 \}
\]

and

\[
l(y) = y^{[2n]} \quad \text{for} \quad y \in D(l).
\]

Corresponding to \(l\) we consider the following operators in \(L^2(a, b)\): The maximal operator \(L\) defined by

\[
D_L = D = \{ y \mid y \in D(l) \cap L^2(a, b), l(y) \in L^2(a, b) \}
\]

and

\[
Ly = l(y) \quad \text{for} \quad y \in D.
\]

\(L_0'\) is the restriction of \(L\) to the set \(D_0'\) of functions in \(D\) with compact support. The minimal operator \(L_0\) is the closure of \(L_0'\), its domain is denoted by \(D_0\). Finally, \(L_c\) will denote a closed extension of \(L_0\), \(D_c\) its domain, \(L_s\) a self-adjoint extension of \(L_0\) and \(D_s\) its domain.

b) The perturbing operators.

The theory of \(L_c\)-compactness requires that \(B\) is a closable operator defined on \(D_c\).
In the following, when \( k = n + r + 1 \) with \( 0 \leq r \leq n - 1 \), we assume, that \( (1/p_0)^{(n)} \) and \( p_r^{(r)} \) exist, loc. a.c., for \( 1 \leq r \leq n - 1 \). This implies, that \( y^{(n+r)} \) exists, loc. a.c., and hence the existence of \( y^{(k)} \) for all \( y \in D \). Then we can define for every complex-valued function \( \beta \) on \((a,b)\) the formal differential operator \( b_k \) by

\[
D(b_k) = D
\]

and

\[
b_k(y) = \beta y^{(k)} \quad \text{for} \quad y \in D(b_k).
\]

If \( b_k(y) \in \mathcal{L}^2(a,b) \) for \( y \in D_c \) and for some fixed \( k \), \( 0 \leq k \leq 2n - 1 \), we define the operator \( B_k' \) by

\[
D_{B_k'} = D_c
\]

and

\[
B_k' y = b_k(y) \quad \text{for} \quad y \in D_{B_k'}.
\]

**Lemma 2.1.** Suppose that \( B_k' \) is defined on \( D_c \), and that \( \beta \) is a.e. equal to a function \( \beta_1 \) with the following properties:

1. \( S = \{ x \mid x \in (a,b), \beta_1(x) = 0 \} \) is closed.
2. For every interval \([\alpha, \beta]\) \( \subset (a,b) \setminus S \), there exists a \( K_{\alpha, \beta} > 0 \) such that \( 1/\beta_1(x) < K_{\alpha, \beta} \) for \( \alpha \leq x \leq \beta \).

Then \( B_k' \) is closable.

This form of the conditions is due to conversations with T. Gamelin.

**Proof.** a) We consider first the case \( \beta(x) \equiv 1 \). Let \( B_{k0} \) be the restriction of \( B_k' \) to functions with compact support. It is easy to prove that \( B_k' \) is contained in \( B_{k0} \) and hence closable.

b) In the general case let

1. \( y_r \to r \to \infty 0 \) in \( \mathcal{L}^2(a,b) \),
2. \( B_k' y_r \to r \to \infty 0 \) in \( \mathcal{L}^2(a,b) \).

From (ii) it follows, that \( z(x) = 0 \) a.e. for \( x \in S \). Also for any interval \([\alpha, \beta]\) \( \subset (a,b) \setminus S \), the conditions (2) and (ii) imply

\[
\int_\alpha^\beta \left( \frac{z(x)}{\beta(x)} - y_r^{(k)}(x) \right)^2 dx \to r \to \infty 0
\]

From a) it follows that \( z(x) = 0 \) a.e. on \([\alpha, \beta]\). Hence \( z = 0 \) in \( \mathcal{L}^2(a,b) \), and the lemma is proved.

In the following we shall assume that \( \beta \) is a.e. equal to a function \( \beta_1 \), having the properties (1) and (2) stated in lemma 2.1, so that \( B_k' \), whenever defined on \( D_c \), is closable.
3. Formulation of the problem.

**Definition 3.1.** For any closed operator $A$ in a Hilbert space $H$ with norm $\| \cdot \|$, we define the $A$-norm of $x \in D_A$ by

$$\|x\|_A^2 = \|x\|^2 + \|Ax\|^2.$$ 

Then $D_A$ is a Hilbert space with the $A$-norm.

**Definition 3.2.** A set $S \subseteq D_A$ is said to be $A$-bounded, if $\|x\|_A < K$ for $x \in S$. An operator $B$ defined on $D_A$ is said to be $A$-defined. When $B$ maps every $A$-bounded set into a bounded set, $B$ is called $A$-bounded. When $B$ maps every $A$-bounded set into a precompact set, $B$ is said to be $A$-compact.

**Remark 3.3.** When $A$ is closed, and $B$ is a closable $A$-defined operator, $B$ is $A$-bounded.

**Lemma 3.4.** $B$ is $A$-compact if, and only if, $B(A - \lambda)^{-1}$ is compact for some $\lambda$ in $\varrho(A)$, the resolvent set of $A$ (or, equivalently, for all $\lambda \in \varrho(A)$).

**Definition 3.5.** $B$ is said to be of $A$-Hilbert–Schmidt type, if $B(A - \lambda)^{-1}$ is a Hilbert–Schmidt operator for some $\lambda \in \varrho(A)$ (for all $\lambda \in \varrho(A)$).

For every $L_c$ and $k$, $0 \leq k \leq 2n - 1$, we shall consider the following problem: For which functions $\beta$ is $B_k' \in L_c$-compact operator, resp. of $L_c$-Hilbert–Schmidt type?

Instead of treating this problem directly we consider the corresponding problem for the quasi-derivatives: Let the operator $B_k$ be defined by

$$B_k y = \beta y^{(k)}.$$ 

For which functions $\beta$ is $B_k$ an $L_c$-defined and $L_c$-compact operator, resp. of $L_c$-Hilbert–Schmidt type?

For $0 \leq k \leq n - 1$ we have $B_k = B_k'$; for $k = n$ and $p_0(x) \neq 0$ a.e. the solution of the problem for $B_k$ can immediately be applied to $B_k'$. For $n + 1 \leq k \leq 2n - 1$ the derivatives $y^{(k)}$ can be expressed linearly by the $y^{(s)}$, $s = 2n - k, \ldots, k$, with certain functions of the $p_r^{(k-n-q)}$, $q = r, \ldots, k - n$, $r = 0, \ldots, k - n$, as coefficients; then the results for the $B_s$ can be applied to $B_k'$, at least to give sufficient conditions on $\beta$ in order that $B_k'$ be $L_c$-compact.

4. Local conditions and reduction of the main problem.

**Lemma 4.1.** In the regular case, i.e. when $(a, b)$ is finite, and $1/p_0$, $p_1, \ldots, p_n$ are integrable on $(a, b)$, for any set of complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_{2n-1}, \beta_0, \beta_1, \ldots, \beta_{2n-1}$, there exists a function $y \in D$ such that
\[ y^{[k]}(a) = \alpha_k, \quad y^{[k]}(b) = \beta_k, \quad k = 0, 1, \ldots, 2n - 1. \]

**Proof.** See Neumark [5, § 17.3, lemma 2].

**Lemma 4.2.** For every \( L \) and \( k \) a necessary condition in order that \( B_k \) be \( L_0' \)-defined is that \( \beta \in L^2_{\text{loc}}(a,b) \), that is

\[ \int_c |\beta(x)|^2 \, dx < \infty \]

for every compact subset \( c \) of \((a,b)\).

**Proof.** By means of lemma 4.1 it is simple to construct for any \( x_0 \in (a,b) \) a function \( y_0 \in D_0' \) such that

\[ y_0^{[k]}(x_0) = 1. \]

From this and the continuity of \( y_0^{[k]} \) the conclusion of the lemma follows.

**Lemma 4.3.** In the regular case \( \beta \in L^2(a,b) \) implies that \( B_k \) is \( L \)-compact for \( k = 0, 1, \ldots, 2n - 1 \).

**Proof.** Let \( y_1, y_2, \ldots, y_{2n} \) be a system of linearly independent solutions of the equation

\[ (l - \lambda)y = 0 \]

for some non-real \( \lambda \), normed such that the Wronskian is 1. Set

\[
(1) \quad v_k(x) = \begin{vmatrix}
  y_1(x) & \ldots & y_{k-1}(x) & y_{k+1}(x) & \ldots & y_{2n}(x) \\
  \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
  y_1^{[2n-2]}(x) & \ldots & y_{k-1}^{[2n-2]}(x) & y_{k+1}^{[2n-2]}(x) & \ldots & y_{2n}^{[2n-2]}(x)
\end{vmatrix}.
\]

Then the solution of the equation

\[ (l - \lambda)z = f \]

and its quasi-derivatives are given by

\[
(2) \quad z^{[k]}(x) = \sum_{i=1}^{2n} c_i y_i^{[k]}(x) + \sum_{i=1}^{2n} y_i^{[k]}(x) \int_0^x v_i(\xi) f(\xi) \, d\xi, \quad k = 0, 1, \ldots, 2n - 1.
\]

Let \( \{z_s\} \) be an \( L \)-bounded sequence. Then by (2) for \( k = 0 \) and Schwarz' inequality

\[ \left\{ \sum_{i=1}^{2n} c_i y_i \right\} \text{ is bounded in } L^2(a,b), \]
and hence

\[ |c_{is}| < K \quad \text{for} \quad i = 1, 2, \ldots, 2n, \ s = 1, 2, \ldots. \]

We choose a subsequence \( \{z_{ni}\} \) such that

\[ c_{i z_{ni}} \xrightarrow{i \to \infty} k_i, \quad i = 1, 2, \ldots, 2n \]

and

\[ f_{z_{ni}} \xrightarrow{i \to \infty} f, \quad \text{weakly}. \]

Then by Lebesgue's dominated convergence-theorem

\[ B_k y_{z_{ni}} \xrightarrow{i \to \infty} \beta(x) \left\{ \sum_{i=1}^{2n} k_i y_i^{[k]}(x) + \sum_{i=1}^{2n} y_i^{[k]}(x) \int_0^x v_i(\xi) f(\xi) d\xi \right\} \]

in \( L^2(a, b) \).

**Definition 4.4.** Unmixed boundary conditions are conditions of the form

\[ \sum_{i=0}^{2n-1} \alpha_i y_i^{[\ell]}(a) = 0 \quad \text{and} \quad \sum_{i=0}^{2n-1} \beta_i y_i^{[\ell]}(b) = 0. \]

**Definition 4.5.** When \( \ell \) is a formal differential operator applied to functions on \( (a, b) \), we shall denote by \( L(\alpha, \beta) \) the maximal operator corresponding to \( \ell \) applied to functions on \( [\alpha, \beta] \subset (a, b) \). If \( L_c \) is defined by certain unmixed boundary conditions at the endpoints \( a \) and \( b \), and \( a < \alpha < \beta < b \), we shall denote by \( L_c(\alpha, \beta) \) the operator corresponding to \( \ell \) applied to functions on \( (a, \alpha) \) and with the same boundary conditions at the point \( a \) as \( L_c \), but with no boundary conditions at \( a \). \( L_c(\beta, b) \) is defined in the same way, and the same notation is used for \( B_k \). If there are no boundary conditions at the point \( a \), then \( L_c(a, \alpha) \) shall mean \( L(a, \alpha) \), and similarly for \( b \).

**Theorem 4.6.** Let the operator \( L_c \) be defined by certain unmixed boundary conditions at the endpoints \( a \) and \( b \). Then a necessary and sufficient condition for the operator \( B_k \) to be \( L_c \)-bounded or \( L_c \)-compact is that

\[ (1) \ \beta \in L^2_{\text{loc}}(a, b) \]

and

\[ (2) \ B_k(a, \alpha) \text{ and } \beta_k(\beta, b) \text{ are } L_c \text{-bounded, resp. } L_c \text{-compact with respect to } L_c(a, \alpha) \text{ and } L_c(\beta, b) \text{ for some } \alpha, \beta \text{ with } a < \alpha < \beta < b \text{ (equivalently, for all such } \alpha, \beta). \]

**Proof.** For every \( L_c \)-bounded sequence \( \{y_n\} \) the restrictions of \( y_n \) to the intervals \((a, \alpha), (\alpha, \beta)\) and \((\beta, b)\) form \( L_c(a, \alpha) \)-bounded, \( L_c(\alpha, \beta) \)-bounded and \( L_c(\beta, b) \)-bounded sequences. From this follows the sufficiency.
By means of lemma 4.1 it is simple to construct to a given $L_c(a, \alpha)$-bounded sequence $\{z_n\}$ an $L_c$-bounded sequence $\{y_n\}$ such that $z_n$ is the restriction of $y_n$ to $(a, \alpha)$, and similarly for the intervals $(x, \beta)$ and $(\beta, b)$. From this follows the necessity.

By theorem 4.6 the problem concerning compactness is reduced to the following main cases:

I. The interval $[0, 1]$ with both endpoints regular, boundary conditions at 0 and no boundary conditions at 1.

II. $L$ and $L_s$ on $[0, \infty)$ with 0 regular.

III. $L$ on $[0, 1)$ with 0 regular, 1 singular.

Remark 4.7. Every theorem concerning $L_c$-compactness of the $B_k$ remains valid if the operator $L_c$ is changed by addition of a bounded function $r$ to $p_n$. For, obviously, a set $S \subset D_c$ is $L_c$-bounded if, and only if, it is $(L_c+r)$-bounded.

5. Investigation of the main cases.

Case I: The interval $[0, 1]$ with both endpoints regular, boundary conditions at 0 and no boundary conditions at 1.

Theorem 5I,1. Suppose that $y^{[k]}(0) = 0$ is not a boundary condition for $L_c$ for some $k$ with $0 \leq k \leq 2n - 1$. Then $\beta \in L^2(0, 1)$ is necessary for $B_k$ to be $L_c$-bounded and sufficient for $B_k$ to be $L_c$-compact. Also $\beta \in L^2(0, 1)$ is sufficient in order that $B_k$ be of $L_s$-Hilbert–Schmidt type.

Proof. a) Suppose that $B_k$ is $L_c$-bounded. The existence of a function $y \in D_c$, such that $y^{[k]}(0) \neq 0$, implies

$$
\int_0^\varepsilon |\beta(x)|^2 \, dx < \infty \quad \text{for some } \varepsilon > 0 ,
$$

and then by lemma 4.2 $\beta \in L^2(0, 1)$.

b) By lemma 4.3 $\beta \in L^2(0, 1)$ implies that $B_k$ is $L_c$-compact.

For any $L_s$ the $c_i$ in the expression (2) for $z^{[k]}$ used in the proof of lemma 4.3 are bounded linear functionals of $f$ for $i = 1, 2, \ldots, 2n$. Then there exist functions $h_i \in L^2(0, 1)$, $i = 1, 2, \ldots, 2n$, such that

$$
c_i(f) = \int_0^1 h_i(\xi) f(\xi) \, d\xi .
$$

By substitution of these expressions in (2) of lemma 4.3 we obtain the following representation of $(B_k - \lambda)^{-1}$:
\[(B_k - \lambda)^{-1}f(x) = \int_0^1 K_k(x, \xi) f(\xi) \, d\xi\]

with
\[
K_k(x, \xi) = \begin{cases}
\beta(x) \sum_{i=1}^{2n} y_i^{[k]}(x)[h_i(\xi) + v_i(\xi)], & 0 \leq \xi \leq x, \\
\beta(x) \sum_{i=1}^{2n} y_i^{[k]}(x)h_i(\xi), & x < \xi.
\end{cases}
\]

Since the \(y_i^{[k]}\) and the \(v_i\) (see lemma 4.3 (1)) are continuous on \([0, 1]\), it follows that \(\beta \in \mathcal{L}^2[0, 1]\) implies
\[
\int_0^1 \int_0^1 |K_k(x, \xi)|^2 \, dx \, d\xi < \infty \quad \text{for} \quad k = 0, 1, \ldots, 2n - 1.
\]

Lemma 5I,2. For \(0 < a \leq \infty\) let \(f \in \mathcal{L}^2(0, a)\) and set
\[
F(x) = \int_0^x f(t) \, dt.
\]

Let \(\psi\) be a complex-valued function on \([0, a]\) and let \(T\) be the operator in \(\mathcal{L}^2(0, a)\) defined by
\[
D_T = \{f \mid f \in \mathcal{L}^2(0, a), \, \psi F \in \mathcal{L}^2(0, a)\}
\]
and
\[
Tf(x) = \psi(x)F(x) \quad \text{for} \quad f \in D_T.
\]

Then \(T\) is a compact operator with \(D_T = \mathcal{L}^2(0, a)\) if, and only if
\[
\psi(x) = \int_0^x |\psi(t)|^2 dt < \infty, \quad 0 < x < a;
\]
\[
x \psi(x) \to_{x \to 0} 0;
\]
\[
x \psi(x) \to_{x \to \infty} 0.
\]

This lemma goes back to a corresponding statement concerning boundedness, which for \(a < \infty\), is due to J. Odhnoff (private communication): \(T\) is a bounded operator with \(D_T = \mathcal{L}^2(0, a)\) if, and only if, (1) holds and
\[
x\psi(x) < K \quad \text{for} \quad 0 < x < a.
\]

The idea of the proof given here is due to E. Thue Poulsen.

Proof. We prove the lemma for \(a = \infty\); for \(a < \infty\) it follows easily (in that case also (3) is trivially implied by (1)).

(a) Suppose that \(T\) is compact with \(D_T = \mathcal{L}^2(0, \infty)\) and consider for \(0 < x < \infty\) the functions \(f_x\) defined by
\[ f_x(t) = \begin{cases} x^{-1} & 0 \leq t \leq x, \\ 0 & x < t < \infty. \end{cases} \]

The family \( \{f_x\}_{0<x<\infty} \) is bounded in \( L^2(0,\infty) \), hence (1) follows. Since \( f_x(t) \to 0 \) for \( x \to 0 \) or \( x \to \infty \) and \( T \) is compact, it follows that
\[ x^2 \Psi(x) \leq \|Tf_x\|^2 \to 0 \quad \text{for} \quad x \to 0 \text{ or } x \to \infty. \]

(b) We now prove that (1), (2) and (3) imply that \( T \) is compact.
(i) Let \( x^2 \Psi(x) \leq c \) for \( 0 < x < \infty \). For \( f \in L^2(0,\infty) \), \( f(x) \geq 0 \), \( 0 < \alpha < \beta < \infty \), we have
\[
\int_\alpha^\beta |\psi(x)|^2 F^2(x) \, dx = -\Psi(\beta) F^2(\beta) + \Psi(\alpha) F^2(\alpha) + 2 \int_\alpha^\beta x^2 \Psi(x) \frac{F(x)}{x} f(x) \, dx.
\]
By Schwarz’ and Hardy’s inequalities
\[
\left| \int_\alpha^\beta |\psi(x)|^2 F^2(x) \, dx \right| \leq \beta \Psi(\beta) \|f\|_2^2 + \alpha \Psi(\alpha) \|f\|_2^2 + 4c \|f\|_2^2
\]
\[ \leq 6c \|f\|_2^2. \]

(For Hardy’s inequality, see e.g. Hardy, Littlewood and Polya [3]). This implies, that \( T \) is a bounded operator with \( D_T = L^2(0,\infty) \), and \( \|T\|^2 \leq 6c \).

(ii) For a function \( \psi \) with compact support satisfying (1) the result follows from Schwarz’ inequality and Lebesgue’s theorem on dominated convergence.

(iii) For any \( \psi \) satisfying (1), (2), and (3), let \( \psi_n = \psi x_n \), where \( x_n \) is the characteristic function of \([1/n, n]\). By (i) and (ii) the corresponding operators \( T_n \) form a sequence of compact operators converging uniformly to \( T \), which proves, that \( T \) is compact.

**Theorem 51.3.** When \( y^{2n-1}(0) = 0 \) is a boundary condition for \( L_c \), and \( p_n \in L^2(0,1) \), a necessary and sufficient condition in order that \( B_{2n-1} \) be \( L_c \)-compact is that
\[
(1) \quad \int_x^1 |\beta(t)|^2 \, dt < \infty \quad \text{for} \quad 0 < x < 1,
\]
\[
(2) \quad x \int_x^1 |\beta(t)|^2 \, dt \to_{x \to 0} 0.
\]

**Proof.** (a) Suppose, that \( \beta \) satisfies (1) and (2). For \( f = L_c y \) we have
$$y^{2n-1}(x) = \int_0^x [p_n(t)y(t) - f(t)] \, dt.$$  

If \( \{y_s\} \) is an \( L_c \)-bounded sequence, then

$$|y(x)| < K \quad \text{for} \quad 0 \leq x \leq 1, \quad s = 1, 2, \ldots.$$  

Hence \( \{p_n y_s - f_s\} \) is bounded in \( L^2(0, 1) \), and by lemma 5I.2 the sequence \( \{\beta y_s^{2n-1}\} \) is compact.  

(b) Let \( B_{2n-1} \) be \( L_c \)-compact and consider a set \( \{y_\varepsilon\}_{0 < \varepsilon \leq 1} \) such that

$$\frac{d}{dx} (y_\varepsilon^{2n-1}(x)) = \begin{cases} \varepsilon^{-1}, & 0 \leq x \leq \varepsilon, \\ 0, & \varepsilon < x \leq 1. \end{cases}$$

It is easy to see, that we can choose \( \{y_\varepsilon\}_{0 < \varepsilon \leq 1} \) to be an \( L_c \)-bounded set, and then by the proof of lemma 5I.2a,

$$\varepsilon \int_\varepsilon^1 |\beta(x)|^2 \, dx \xrightarrow{\varepsilon \to 0} 0.$$  

**Theorem 5I.4.** If \( y^{k+1}(0) = 0 \) are boundary conditions for \( L_c \) for some \( k, 0 \leq k \leq n - 2 \), and \( v = 0, 1, \ldots, p \) with \( 0 \leq p \leq n - 2 - k \), and \( y^{k+p+1}(0) = 0 \) is not a boundary condition for \( L_c \), then it is necessary for \( B_k \) to be \( L_c \)-bounded and sufficient for \( B_k \) to be \( L_c \)-compact, that

$$\beta(x)x^{p+1} \in L^2(0, 1).$$

**Proof.** By means of the expression (2) of lemma 4.3 for \( y^{k+p+1} \) the proof is straightforward.

**Remark 5I.5.** Similar, more complicated relations hold for \( n - 1 \leq k \leq 2n - 2, \ 0 \leq p \leq 2n - 2 - k \).

**Case II:** \( L \) and \( L_s \) on the interval \([0, \infty]\) with the endpoint 0 regular.

**Theorem 5II.1.** Let \( A(x) \) be an \( n \times n \) matrix, whose coefficients are complex-valued functions on \([0, \infty)\) such that for sufficiently large \( x \), say \( x > x_0 \),

$$A(x) = A_0(x) + A_1(x),$$

where the elements of \( A_0(x) \) are loc. a.c., and the elements of \( A_0'(x) \) and \( A_1(x) \) are integrable on \((x_0, \infty)\). Let

$$w_1(x), w_2(x), \ldots, w_n(x)$$

be the eigenvalues of \( A_0(x) \) arranged such that \( w_i(x) \) is continuous for \( i = 1, 2, \ldots, n \) and suppose further, that
\[
\lim_{x \to \infty} \text{Re} \left( w_i(x) - w_k(x) \right) = 0 \quad \text{for} \quad i \neq k, \; i, k = 1, 2, \ldots, n.
\]

Then the system of equations
\[
\frac{dy}{dx} = A(x)y(x)
\]
has \(n\) linearly independent solutions \(y_j, \; j = 1, 2, \ldots, n\), such that
\[
y_{ik}(x) = c_{ij}(x) \exp \left( \int_{0}^{x} w_k(\xi) d\xi \right),
\]
where
\[
c_{jk}(x) \xrightarrow{k \to \infty} c_{jk}.
\]

**Proof.** We refer to Neumark [5, §22.1, Satz 2, Folgerung 1].

**Theorem 5III,2.** Suppose that the coefficients of \(l\) satisfy the conditions
\[
1/p_0(x) = a_0(x) + b_0(x), \quad p_i(x) = a_i(x) + b_i(x), \quad i = 1, 2, \ldots, n,
\]
such that for \(x > x_0\) the functions \(a_i\) are loc. a.c.,
\[
\int_{x_0}^{\infty} |a_i'(x)| dx < \infty \quad \text{and} \quad \int_{x_0}^{\infty} |b_i(x)| dx < \infty
\]
for \(i = 0, 1, \ldots, n\), and
\[
\lim_{x \to \infty} a_0(x) = 0.
\]

Then, for \(k = 0, 1, \ldots, 2n - 1\), \(\beta \in \mathcal{L}^2(0, \infty)\) implies that \(B_k\) is \(L\)-compact and that \(B_k\) is of \(L_s\)-Hilbert–Schmidt type.

If \(y^{[k]}(0) = 0\) is not a boundary condition for \(L_s\), then \(\beta \in \mathcal{L}^2(0, \infty)\) is also a necessary condition in order that \(B_k\) be of \(L_s\)-Hilbert–Schmidt type.

**Proof.** (a) Suppose, that \(\beta \in \mathcal{L}^2(0, \infty)\). The equation
\[
l(y) - \lambda y = 0
\]
is equivalent to a system
\[
\frac{d\tilde{y}}{dx} = A(x)\tilde{y}(x),
\]
where
\[
\tilde{y} = (y, y^{[1]}, \ldots, y^{[2n-1]})
\]
and
\[
A(x) = A_0(x) + A_1(x)
\]
with

\[
A_0(x) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \cdots & \ddots & \vdots \\
0 & a_{n-1} & \cdots & \cdots & a_1 & 0 \\
0 & 0 & \cdots & a_n & 0 & -1 \\
an - \lambda & 0 & \cdots & 0 & 0 & 0 
\end{pmatrix}
\]

\((a_0)\) is in the \(n'th\) row and \((n+1)'th\) column of the \(n \times n\) matrix \(A_0(x)\). The elements of \(A_0'\) exist and are integrable on \((x_0, \infty)\), and the coefficients of \(A_1\) are integrable on \((x_0, \infty)\). The eigenvalues \(w_1(x), \ldots, w_{2n}(x)\) of \(A_0(x)\) are the roots of the equation

\[
\frac{1}{a_0(x)}e^{2n} - a_1(x)e^{2n-2} + a_2(x)e^{2n-4} - \ldots + (-1)^n(a_n(x) - \lambda) = 0.
\]

We choose \(\lambda\) such that

\[
\lim_{x \to \infty} \text{Re}(w_i(x) - w_k(x)) \neq 0 \quad \text{for} \quad i \neq k, \quad i, k = 1, 2, \ldots, n.
\]

Let \(r_i(x) = \text{Re}w_i(x), i = 1, 2, \ldots, 2n\), and choose the order of the \(w_i\) such that

\[
r_1(\infty) < r_2(\infty) < \ldots < r_{2n-1}(\infty) < r_{2n}(\infty).
\]

Then there exist \(x_1 > 0\) and \(\varepsilon > 0\) such that

\[
r_1(x) < r_2(x) < \ldots < r_n(x) < -\varepsilon < 0 < \varepsilon < r_{n+1}(x) < \ldots < r_{2n}(x)
\]

for \(x > x_1\), and

\[
r_i(x) = -r_{2n-i+1}(x), \quad i = 1, 2, \ldots, n.
\]

Application of theorem 5II.1 to (2) shows, that (1) has \(2n\) linearly independent solutions \(y_1, \ldots, y_{2n}\) such that

\[
y_k^{(j)}(x) = c_{jk}(x) \exp \left( \int_0^x w_k(\xi) \, d\xi \right) = c_{jk}(x)W_k(x),
\]

where

\[
c_{jk}(x) \xrightarrow{x \to \infty} c_{jk} \quad \text{for} \quad j = 1, \ldots, 2n, \quad k = 0, \ldots, 2n - 1,
\]

and
\[ W_k(x) = \exp \left( \int_0^x w_k(\xi) \, d\xi \right), \quad k = 1, 2, \ldots, 2n. \]

We choose \( \lambda \) such that \( c_{jk} \neq 0 \) for \( j = 1, \ldots, 2n, \ k = 0, \ldots, 2n - 1. \) Then
\[
y_i \in L^2(0, \infty) \quad \text{for} \quad i = 1, \ldots, n, \quad y_i \notin L^2(0, \infty) \quad \text{for} \quad i = n + 1, \ldots, 2n.
\]

Let \( y_1, \ldots, y_{2n} \) be normed such that the Wronskian is 1. Then for \( f \in L^2(0, \infty) \) the solution of the equation
\[ Ly - \lambda y = f \]

together with its derivatives are given by
\[
y^{[k]}(x) = \sum_{i=1}^{n} k_i y^{[k]}_i(x) + \sum_{i=1}^{n} y^{[k]}_i(x) \int_0^x v_i(\xi)f(\xi) \, d\xi - \sum_{i=n+1}^{2n} y^{[k]}_i(x) \int_x^\infty v_i(\xi)f(\xi) \, d\xi
\]
for \( k = 0, 1, \ldots, 2n - 1, \) where

\[
v_i(\xi) = (-1)^i c_{0,1}(\xi) W_1(\xi) \ldots c_{0,i-1}(\xi) W_{i-1}(\xi) c_{0,i+1}(\xi) W_{i+1}(\xi) \ldots c_{0,2n}(\xi) W_{2n}(\xi)
\]

\[
= c_i(\xi) W_i(\xi) = c_i(\xi) \exp \left( \int_0^\xi w_i(t) \, dt \right),
\]

and

\[ c_i(\xi) \xrightarrow{\xi \to \infty} c_i, \]

\( i = 1, 2, \ldots, 2n. \) (For the definition of the \( v_i, \) see lemma 4.3 (1).) By the choice of \( \lambda \) we obtain \( c_i \neq 0 \) for \( i = 1, \ldots, 2n. \)

Substitution of the asymptotic expressions for \( y^{[k]}_i \) and \( v_i \) in (4) gives

\[
y^{[k]}(x) = \sum_{i=1}^{n} k_i c_{ki}(x) W_i(x) + \sum_{i=1}^{n} c_{ki}(x) W_i(x) \int_0^x c_i(\xi) W_i(\xi)f(\xi) \, d\xi - \sum_{i=n+1}^{2n} c_{ki}(x) W_i(x) \int_x^\infty c_i(\xi) W_i(\xi)f(\xi) \, d\xi
\]
for \( k = 0, 1, \ldots, 2n - 1. \) Then

\[
B_k y(x) = \beta(x) \sum_{i=1}^{n} k_i c_{ki}(x) W_i(x) + \int_0^\infty K_k(x, \xi) f(\xi) \, d\xi
\]

\[ = B_{k1} y(x) + B_{k2} y(x) \]
where

\[
K_k(x, \xi) = \left\{ \begin{array}{ll}
\beta(x) \sum_{i=1}^{n} c_{ki}(x) c_i(\xi) \exp \left( \int_{\xi}^{x} w_i(t) \, dt \right), & 0 \leq \xi \leq x, \\
-\beta(x) \sum_{i=n+1}^{2n} c_{ki}(x) c_i(\xi) \exp \left( -\int_{x}^{\xi} w_i(t) \, dt \right), & x < \xi.
\end{array} \right.
\]

Since \( |c_{ki}(x)| < K \) for \( 0 \leq x < \infty, i = 1, 2, \ldots, 2n, k = 0, 1, \ldots, 2n-1 \), we have

\[
\int_{x_1}^{x} |c_{ki}(x)|^2 |c_i(\xi)|^2 \exp \left( 2 \int_{x_1}^{\xi} r_i(t) \, dt \right) \, d\xi \leq K^4(2\varepsilon)^{-1} \int_{x_1}^{x} (-2r_i(\xi)) \exp \left( 2 \int_{x_1}^{\xi} r_i(t) \, dt \right) \, d\xi < K_1
\]

for \( i = 1, 2, \ldots, n \), and similarly, for \( i = n+1, \ldots, 2n, k = 0, \ldots, 2n-1 \),

\[
\int_{x_1}^{\infty} |c_{ki}(x)|^2 |c_i(\xi)|^2 \exp \left( -2 \int_{x}^{\xi} r_i(t) \, dt \right) \leq K^4(2\varepsilon)^{-1} \int_{x}^{\infty} 2r_i(\xi) \exp \left( -2 \int_{x}^{\xi} r_i(t) \, dt \right) \, d\xi < K_1.
\]

Also \( \int_{0}^{\infty} |K_k(x, \xi)|^2 \, d\xi < K |\beta(x)|^2 \) for \( 0 \leq x \leq x_1 \) and hence

\[
\int_{0}^{\infty} \int_{0}^{\infty} |K_k(x, \xi)|^2 \, d\xi \, dx \leq K \int_{0}^{\infty} |\beta(x)|^2 \, dx < \infty,
\]

and \( B_{k2} \) is a Hilbert–Schmidt operator for \( k = 0, \ldots, 2n-1 \). From the expression (5) for \( y = y^{[0]} \) it is seen, that if \( \{y_{i,e}\} \) is an \( L \)-bounded sequence, then \( \{k_{i,e}\} \) is bounded for \( i = 1, \ldots, n \). Since

\[
|c_{ki}(x) \exp \left( \int_{0}^{x} r_i(t) \, dt \right)| < K \quad \text{for} \quad 0 \leq x < \infty, \ i = 1, \ldots, n,
\]

it follows from Lebesgue’s theorem on dominated convergence, that \( B_{k1} \) is \( L \)-compact. Thus, we have proved that \( B_k \) is \( L \)-compact.

For any \( L \) the \( k_i \) in (6) considered as functions of \( f \) are bounded linear functionals on \( L^2(0, \infty) \). Hence there exist functions \( h_i \in L^2(0, \infty) \), \( i = 1, \ldots, n \), such that

\[
B_k(L - \lambda)^{-1} f(x) = \int_{0}^{\infty} H_k(x, \xi) f(\xi) \, d\xi,
\]

where

\[
H_k(x, \xi) = \left\{ \begin{array}{ll}
\beta(x) \sum_{i=1}^{n} \left( h_i(\xi) \exp \left( \int_{0}^{\xi} w_i(t) \, dt \right) + c_i(\xi) \right) c_{ki}(x) \exp \left( \int_{\xi}^{x} w_i(t) \, dt \right), & 0 \leq \xi \leq x, \\
\beta(x) \sum_{i=1}^{n} h_i(\xi) c_{ki}(x) \exp \left( \int_{0}^{\xi} w_i(t) \, dt \right) - \sum_{i=n+1}^{2n} c_{ki}(x) c_i(\xi) \exp \left( -\int_{x}^{\xi} w_i(t) \, dt \right), & x < \xi.
\end{array} \right.
\]
It is evident that, when \( \beta \in \mathcal{L}^2(0, \infty) \), the terms involving \( h_i(\xi) \) are also square-integrable, so that \( B_k(L_s - \lambda)^{-1} \) is a Hilbert–Schmidt operator.

(b) Suppose now that
\[
\int_0^\infty \int_0^\infty |H_k(x, \xi)|^2 \, d\xi \, dx < \infty
\]
for some \( k \), \( 0 \leq k \leq 2n - 1 \), and that \( y^{(k)}(0) = 0 \) is not a boundary condition for \( L_s \). We choose \( x_0 \) and \( x_1 \) such that \( 0 < x_1 < x_0 \) and, for some \( \delta > 0 \),
\[
r_n(x) > r_n(\infty) - \delta > r_{n-1}(\infty) + \delta > r_i(x) \quad \text{for} \ x > x_0, \ i = 1, 2, \ldots, n-1,
\]
and
\[
|c_n(\xi)c_{kn}(x)| - \sum_{i=1}^{n-1} |c_i(\xi)c_{ki}(x)| \exp(-2\delta x_1) > K > 0 \quad \text{for} \ x > x_0.
\]
This is possible, since all the \( c_i \) and \( c_{ki} \) have finite limit values different from 0 as \( x \to \infty \). Then, for \( x > x_0 + x_1 \),
\[
\int_0^x \left( \sum_{i=1}^n c_i(\xi)c_{ki}(x) \exp\left( \int_\xi^x w_i(t) \, dt \right) \right)^2 \, d\xi \geq K^2 \int_{x_0}^{x-x_1} \exp\left( 2 \int_\xi^x r_n(t) \, dt \right) \, d\xi > K_1 > 0.
\]
Finally we can find \( x_2 > x_0 + x_1 \) such that
\[
\int_0^x \left( \sum_{i=1}^n h_i(\xi)c_{ki}(x) \exp\left( \int_\xi^x w_i(t) \, dt \right) \right)^2 \, d\xi < K_1/4 \quad \text{for} \ x \geq x_2.
\]
Then
\[
\int_0^\infty |H_k(x, \xi)|^2 \, d\xi > |\beta(x)|^2 K_1/4 \quad \text{for} \ x \geq x_2,
\]
hence
\[
\int_{x_2}^\infty |\beta(x)|^2 \, dx < \infty.
\]
Also by theorem 4.6 and theorem 51,1
\[
\int_{x_2}^\infty |\beta(x)|^2 \, dx < \infty,
\]
and we have proved, that \( \beta \in \mathcal{L}^2(0, \infty) \).

**Theorem 5II,3.** If, on the interval \((-\infty, \infty)\), the coefficients satisfy the asymptotic relations required in theorem 5II,2 both for \( x \to \infty \) and for \( x \to -\infty \), then the index of deficiency is \((0,0)\), and \( L \) is self-adjoint. A
necessary and sufficient condition in order that $B_n$ be of $L_s$-Hilbert–Schmidt type is that $\beta \in L^2(\mathbb{R})$.

**Proof.** The proof is similar to the proof of theorem 5II.2.

**Remark 5II.4.** For $p_0 = 1$, $p_i = 0$ for $i = 1, 2, \ldots, n$, the result of Agudo and Wolf [1] follows.

**Theorem 5II.5.** The following conditions are sufficient in order that $B_{2n-1}$ be $L$-compact:

1. $\beta \in L^2(0, \infty)$
2. $\beta(x) \left( \int_0^x |p_n(t)|^2 \, dt \right)^{\frac{1}{2}} \in L^2(0, \infty)$
3. $x \int_x^\infty |\beta(t)|^2 \, dt \to 0$ as $x \to \infty$

**Proof.** For $f = Ly$,

$$B_{2n-1} y(x) = \beta(x) y^{2n-1}(0) - \beta(x) \int_0^x f(t) \, dt + \beta(x) \int_0^x p_n(t) y(t) \, dt$$

$$= B'_{2n-1} y(x) + B''_{2n-1} y(x) + B'''_{2n-1} y(x) .$$

Since $\{y_s^{2n-1}(0)\}$ is bounded for an $L$-bounded sequence $\{y_s\}$, it follows from (1) that $B'_{2n-1}$ is $L$-compact. By lemma 5I.2, conditions (1) and (2) imply that $B''_{2n-1}$ is $L$-compact. By lemma 4.3 the operator $B_{2n-1}(0,K)$ is $L(0,K)$-compact, and since this evidently holds for $B'_{2n-1}(0,K)$ and $B''_{2n-1}(0,K)$, also $B'''_{2n-1}(0,K)$ is $L(0,K)$-compact. Then an $L$-bounded sequence $\{y_s\}$ has a subsequence $\{y_{s_k}\}$ such that

$$B''_{2n-1} y_{s_k}(x) \to z(x) \quad \text{a.e. on } (0,K) .$$

Since this holds for any $K > 0$, there exists a subsequence $\{y_{s_l}\}$ such that

$$B'''_{2n-1} y_{s_l}(x) \to z(x) \quad \text{a.e. on } (0,\infty) .$$

By Lebesgue’s dominated convergence theorem, this together with (3) implies

$$B'''_{2n-1} y_{s_l} \to z \quad \text{in } L^2(0,\infty) ,$$

and $B'''_{2n-1}$ is $L$-compact.

**Theorem 5II.6.** We consider the case, where $n = 1$, $p(x) = p_0(x) \geq 0$, $1/p \in L^1(0,\infty)$, and
\[ xP(x) = x \int_{x}^{\infty} \frac{dt}{p(t)} < K \quad \text{for} \quad 0 \leq x < \infty \quad \text{and} \quad p_1(x) \equiv 0. \]

Then \( B_1 \) is \( L \)-compact if, and only if, \( \beta \) satisfies the conditions

1. \( \beta \in \mathcal{L}^2(0, \infty), \)

2. \( x \int_{x}^{\infty} |\beta(t)|^2 \, dt \to x \to \infty 0 \)

Proof. By theorem 5II.5 conditions (1) and (2) imply that \( B_1 \) is \( L \)-compact. Suppose, on the other hand, that \( B_1 \) is \( L \)-compact. Define the functions \( y_a \) for \( 0 < a < \infty \) by

\[
y_a(x) = \begin{cases} 
a^{-1}xP(x) + a^{-\frac{1}{2}} \int_{x}^{a} P(t) \, dt - \frac{a^{-\frac{1}{2}} \int_{0}^{a} P(t) \, dt}{P(0)} P(x), & 0 \leq x \leq a, \\
a^{\frac{1}{2}}P(x) \left( 1 - \frac{a^{-\frac{1}{2}} \int_{0}^{a} P(t) \, dt}{P(0)} \right), & a < x < \infty. \end{cases}
\]

Then

\[
Ly_a(x) = \begin{cases} 
a^{-1}, & 0 \leq x \leq a, \\
0, & a < x < \infty; \end{cases}
\]

\[
B_1 y_a(x) = \begin{cases} 
\beta(x) \left( a^{-\frac{1}{2}} \int_{0}^{a} P(t) \, dt - a^{-\frac{1}{2}}x \right), & 0 \leq x \leq a, \\
\beta(x) a^{\frac{1}{2}} \left( -1 + \frac{a^{-\frac{1}{2}} \int_{0}^{a} P(t) \, dt}{P(0)} \right), & a < x < \infty. \end{cases}
\]

It follows from simple inequalities that \( \{y_a\}_{a_0 < a < \infty} \) is an \( L \)-bounded set for some \( a_0 > 0 \), and

\[ B_1 y_a(x) \xrightarrow{a \to \infty} 0 \quad \text{pointwise}. \]

This together with the \( L \)-compactness of \( B_1 \) implies

\[ \|B_1 y_a\|_2 \xrightarrow{a \to \infty} 0, \]

and since

\[ a^{-\frac{1}{2}} \int_{0}^{a} P(t) \, dt \xrightarrow{a \to \infty} 0, \]

(1) and (2) follow.

Case III: \( L \) on the interval \([0,1]\) with 0 regular and 1 singular.

We suppose that \( p_n \in \mathcal{L}^2(0,1) \) and consider only the operator \( B_{2n-1} \). The following theorems can be proved by the methods used in I and II.
Remark 5III,1. For every \( y \in D \) the function \( y^{2n-1}(x) \) has a limit as \( x \to 1 \).

Theorem 5III,2. If \( \beta \in L^2(0,1) \), then \( B_{2n-1} \) is \( L \)-compact.

Theorem 5III,3. If there exists a \( y \in D \) such that \( y^{2n-1}(1) \neq 0 \), then \( \beta \in L^2(0,1) \) is necessary in order that \( B_{2n-1} \) be \( L \)-bounded.

Corollary 5III,4. If \( n = 1 \), \( p_1 \equiv 0 \) and \( \int_0^1 dt/p_0(t) \in L^2(0,1) \), then \( \beta \in L^2(0,1) \) is necessary in order that \( B_1 \) be \( L \)-bounded and sufficient in order that \( B_1 \) be \( L \)-compact.

Theorem 5III,5. If \( p_1(x) \equiv 0 \) and \( y^{2n-1}(0) = 0 \) for all \( y \in D \), then \( B_{2n-1} \) is \( L \)-compact if
\[
\varepsilon \int_0^{1-\varepsilon} |\beta(x)|^2 \, dx \to 0.
\]

Corollary 5III,6. If \( n = 1 \), \( p_1(x) \equiv 0 \), \( p_0(x) \geq 0 \) and \( \int_0^1 dt/p_0(t) \notin L^2(0,1) \), \( B_1 \) is \( L \)-compact, if
\[
\varepsilon \int_0^{1-\varepsilon} |\beta(x)|^2 \, dx \to 0.
\]

Theorem 5III,7. If \( n = 1 \), \( p_1(x) \equiv 0 \), \( p_0(x) \geq 0 \), \( P(x) = \int_0^x dt/p_0(t) \notin L^2(0,1) \) and \( (1-x)P(x) < K \) for \( 0 \leq x < 1 \), then \( B \) is \( L \)-compact if, and only if,
\[
\varepsilon \int_0^{1-\varepsilon} |\beta(x)|^2 \, dx \to 0.
\]

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