BRIEF PROOF OF A THEOREM OF BAXTER

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In [1] Baxter proved a theorem that can be stated as follows.

**Theorem.** Let \( X \) be a commutative Banach algebra with identity \( e \). Suppose that \( P \) is a bounded linear transformation on \( X \) which, for some fixed \( b \in X \), satisfies

\[
2P(x \cdot Px) = (Px)^2 + P(bx^2)
\]

for all \( x \in X \), or, equivalently

\[
P(x \cdot Py + y \cdot Px) = (Px)(Py) + P(bxy) .
\]

Then, for given \( f \in X \) and sufficiently small complex \( z \), the equation

\[
g = e + zP(fg)
\]

has the unique solution

\[
g = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} P(b^{n-1}f^n) .
\]

Baxter's proof was heavily combinatorial. It is the purpose of this note to present a brief proof, of a more analytical character. Before proceeding to the proof we remark that the equivalence of (1) and (2) follows trivially upon applying (1) separately to \( x \), \( y \), and \( x + y \).

**Proof of the Theorem.** The key idea is to exploit the differential equations (and the initial condition \( g(0) = e \)) implied by (3) and (4).

Clearly (3) has a unique solution \( g = g(z) \) for all small \( z \); moreover \( g(z) \) is analytic in \( z \). Hence (3) may be differentiated, with the result

\[
g' - zP(fg') = P(fg)
\]

where \( g' = dg(z)/dz \). Now, if \( f \) and \( g \) are given and \( z \) is small, then the equation

\[
h - zP(fh) = P(fg)
\]

has a unique solution \( h \). Since (6) is linear, \( h \) depends linearly on \( g \); indeed, \( h \) has the form

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Received July 31, 1962.

1 Supported by U.S. National Science Foundation Contract G 19117.
\[ h = kg \]

with \( k \in \mathcal{K} \) depending on \( z \) and \( f \) but not on \( g \). Momentarily leaving aside the proof of (7) and the identification of \( k \) we note that (5), (6) and (7) imply \( g'(z) = k(z)g(z) \), and hence

\[ g(z) = \exp \int_0^z k(t) \, dt . \]

We claim now that if \( u \) is defined by

\[ u = \sum_{0}^{\infty} (zbf)^n , \]

then \( k = P(fu) \), i.e., that \( h = gP(fu) \) satisfies (6). We must verify the identity

\[ gP(fu) - zP[fgP(fu)] = P(fg) . \]

Using (2) with \( x = fg \), \( y = fu \), the left member of (10) becomes

\[ gP(fu) + zP[fuP(fg)] - zP(fg)P(fu) - zP(bgfufu) ; \]

on substituting \( zP(fg) = g - c \) from (3) and \( zbfu = u - c \) (which follows from (9)), the quantity (11) is transformed into

\[ gP(fu) + P[fg(g - c)] - (g - c)P(fu) - P(fg(u - c)) \]

which equals \( P(fg) \). This proves (10), (7), and hence (8). Carrying out the indicated integration completes the proof of (4).

REFERENCE